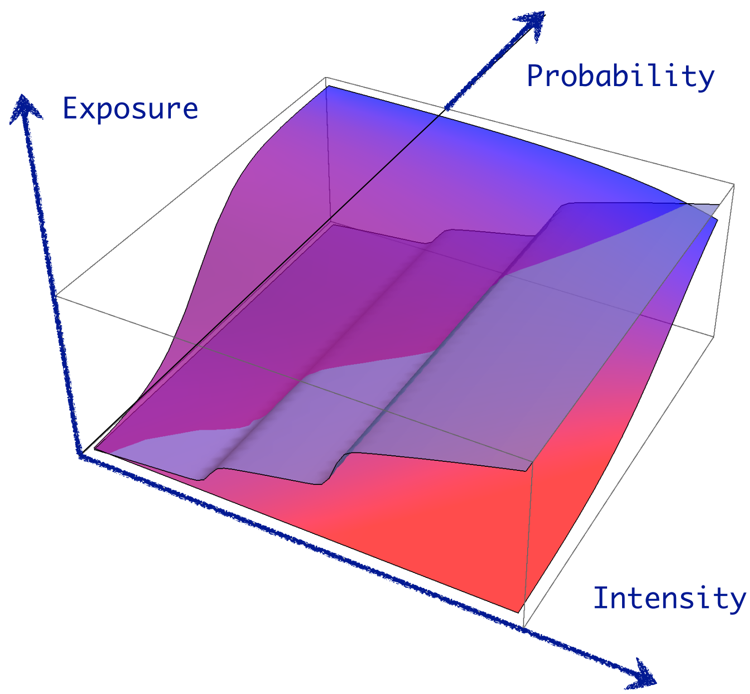
Tuning Meteorological Warning Systems

Jacques Ambühl

29.03.2022



Abstract

A theoretical analysis of the relationship between a meteorological warning system and the addressees of the warnings is presented. Addressees might for example be farmers, hydrological services, civil and airport authorities or management boards of power plants.

A dualistic approach to the relationship between issuer and addressee is favoured in establishing a connexion between the intensity of weather events and their probability of occurence, leading to a definition of the exposure - or vulnerability - of both players.

Theoretical in its essence, this essay is conceived as a formal geometry describing warning decision processes. It is aimed at clarifying some concepts and ideas that have been implemented in previous works and systems.

Contents

1 Introduction 4

2 Aim 5

3 Basic definitions 5

4 Probabilistic formulation of the risk 7
4.1 Probabilistic Hit Rate and False Alarm Rate 9
4.2 Relative Operation Characteristic 13

5 Economic function in the $\{H, F\}$ frame 13

6 Addressee's profile 15
6.1 ROC: $\frac{\partial_Q H}{\partial_Q F}|_{ROC}$ 16
6.2 Economic: $\frac{\partial H}{\partial F}|_{Eco}$ 16
6.3 Synthesis: 17

7 Issuer's profile 19
7.1 Probabilistic warning decision 19
7.2 Probabilistic Hit Rate and False Alarm Ratio 21
7.3 Interlude: trade-off between risk and reward 25

8 Optimal decision, first approach 26
8.1 Economic function in the $\{H, Far\}$ frame 26
8.2 The addressee's perspective 29
8.3 Alternative coordinate system 33

9 Efficiency of a warning system 34

10 Optimal decision, second approach 35
10.1 The issuer's perspective 36

11 Intensity, Probability, Exposure 41
11.1 Addressee and issuer 44
11.2 Inclusion of the efficiency 45
11.3 Creatio ex nihilo: virtual addressee 49
11.4 Performance improving with intensity 50
11.5 Forecaster and freewill 52

12 Modified Climate 52

13 Conclusions **55**

14 Annexes **58**

14.1 Derivation of the economic function 58

14.1.1 Derivation of $M_{A(H,Far)}$ 58

14.1.2 Derivation of $M_{R(H,F)}$ 59

14.2 The Implicit Function Theorem 60

14.3 Integration of a differential equation 60

14.4 Exposures 61

14.4.1 Derivation of $\frac{1}{1+\frac{v_{(p)}}{u_{(p)}}} = E_{(Q)}$ 61

14.4.2 Derivation of $E_{(p)} = \frac{u_{(p)}}{u_{(p)}+v_{(p)}}$ 61

15 References **63**

.

1 Introduction

A theoretical analysis of the relationship between a warning system and the addressees of the warnings is presented. The warning organisation is usually a weather service. Addressees might for example be farmers, hydrological services, civil and airport authorities or management boards of power plants. Beside the addressee, the warning organisation will be referred to as the issuer in the sequel. They will be referred to as "the actors" when considered together.

The programme is realized by establishing a clear distinction between both actors.

The addressee is characterized by a risk and an economic profile. His risk profile takes into account the disasters he might be confronted to (e.g. lost of a harvest in case of precipitation) and the climatology of the triggering meteorological events (e.g. the probability distribution function of precipitation). Disaster risks and meteorological events are merged in a Relative Operating Characteristic. His economic profile is inspired by the classical work by D. S. Richardson, [2 3]. It is defined as the triad formed of the loss resulting in an unpredicted event for which neither mitigating nor protective actions were taken, the cost induced by the protective actions taken in case of occurrence as well as in case of non occurrence of an event, and finally the residual cost occurring in the case of a well predicted event for which protective measures were adequately taken. These various risks and economic factors being fairly entangled, an objective of this work is to provide, at least at theoretical level, some clarity in that matter.

The issuer is expected to fond his warnings on probabilistic forecasts emanating from an Ensemble Prediction System (EPS), or on any system delivering a diagnostic expressed in term of probabilities. The performance of the issuer is then characterized by a dedicated Relative Operating Characteristic (ROC).

The adequate tuning of the warning system requires the rational behaviour of both actors. It enables the minimization of the losses inflicted to the addressee by adverse weather events through the determination of optimal warning thresholds. The dualistic approach proposed provides a quantitative relationship between the performance objectives having to be reached by the issuer and their efficiency, or impact onto the addressee's business. This approach satisfies the requirements formulated in the realm of New Public Management projects undertaken in several weather services. It allows the settlement of service level agreements between both actors.

2 Aim

The interaction occurring between both actors, as well as the their respective roles, is the subject of this essay. Their rational handling - a requirement almost bearing the status of an axiom - shapes the whole reasoning framework.

A first intermediary report addressing these issues has been published in 2010 under [5]. Some of the concepts presented below have been introduced in a popular way in the following (french & german) [6 and 7] publications. The methodology has found applications within the frame of the Swiss Federal project OWARNA (Optimierung der Warnung und Alarmierung vor Naturgefahren bei MeteoSchweiz), [8]. Further works inspired by this approach and making use of genetic algorithms and bayesian statistics are presented in [9].

Theoretical in its essence, this essay is conceived as a formal geometry describing warning decision processes. It is aimed at clarifying some concepts and ideas that have been implemented in the aforementioned works and systems without explicit documentation. Practical applications are mostly disregarded.

3 Basic definitions

An ***event*** is a meteorological phenomena occurring at a given location (a point or a geographical area of finite extension) during a closed time interval. An event is quantified in physical units. Its occurrence - or non occurrence - can be certified *a posteriori* making use of observations. A ***warning*** (or an alarm) is a non probabilistic message dispatched by the issuer to the addressee, foreboding the occurence of an event at the given location during the time interval. A ***disaster*** is any kind of (negative) impact of an event on the addressee's business.

The classical contingency matrix presented below in Table 1. exhibits the four standard warning occurrences related to meteorological events: a) correct rejection: neither did an event occur nor was an alarm raised, b) missed event: an event occurred without alarm, c) false alarm: an alarm was raised although no event occurred and d) hit: an event occurred for which an alarm was correctly raised. The four figures a, b, c, d represent the number of cases accumulated in each category during an assessment period.

	Event			Table 1
Warning	did not occur	did occur	Sum	
issued	c	d	c+d	
not issued	a	b	a+b	
Sum	a+c	b+d	a+b+c+d	

The following arithmetic scores are well known and easily computed:

Hit Rate (also named Probability of Detection) $H = \frac{d}{b+d}$.

False Alarm Rate (also called Probability of False Detection) $F = \frac{c}{a+c}$.

False Alarm Ratio (no other name) $Far = \frac{c}{c+d}$.

Frequency of occurrence of the event: $\Omega = \frac{b+d}{a+b+c+d}$

Frequency of alarms : $\alpha = \frac{c+d}{a+b+c+d}$

Frequency Bias $FB = \frac{\alpha}{\Omega} = \frac{c+d}{b+d} = \frac{H}{1-Far}$ (easily proved).

The hit rate expresses the ratio between the number of events for which a warning was issued and the total number of events. Measuring the overall success of the warning system, it is of paramount importance for both actors.

The false alarm rate and the false alarm ratio conceal a subtle difference. The former measures the frequency at which the addressee's business, instead of running smoothly, is impeded by protective actions taken under fair weather conditions. The latter is expressed as the ratio between the number of mistakenly issued warnings and the total number of issued warnings. It provides a measure of the quality of the service delivered by the issuer. The false alarm rate is an addressee's concern, the false alarm ratio an issuer's issue.

The frequency bias expresses the ratio between the number of warnings delivered during a period of time and the number of events that actually occurred during this period. Does an addressee want to receive more warnings than the number of events he experiences, he will be "over-warned" and might be dubbed "risk adverse". On the contrary, if he happens to be satisfied with less warnings than the number of events he is confronted to, he is "under-warned" and should accordingly be bestowed "risk friendly". This intuitive depiction will be improved along this work.

All these properties are illustrated in simple numerical terms in the following example:

	Event			Table 2
Warning	did not occur	did occur	Sum	
issued	36	64	100	
not issued	254	11	265	
Sum	290	75	365	

The addresses's business being impeded by superfluous protective actions 36 of 290 fair days over the year, the False Alarme Rate is $F = \frac{36}{290} = 0.124$. The quality of the service delivered by the issuer is expressed in terms of the number of mistakenly issued warnings divided by the total number of warnings. With $Far = \frac{36}{100} = 0.36$ in this example, it is not outstanding. The frequency bias expresses the ratio between the number of alarms issued and the number of events that actually occurred. In our example, $FB = \frac{100}{75} = 1.33$, meaning that the user is confronted to 33% more warnings than events. With $H = \frac{64}{75} = 0.853$, the hit rate is fairly good.

As second example, MeteoSwiss is managed according to the tenets of the New Public Management and is required to establish a Performance Related Mandate - "Leistungsauftrag" - with its Supervisory Authority, the Federal Department of Home Affairs. It was specified in the 2014-16 mandate that the hit rate should lie above 85% and the False Alarme Ratio below 30%, thus settling a frequency bias of 1.21 at these boundaries.

A comprehensive description of the verification practice applied at ECMWF may be found in [4].

4 Probabilistic formulation of the risk

Instead of working with *a posteriori* computed frequencies, it is worthwhile to consider probabilistic functions representing the expected distributions of the events taken into consideration. Two of them are introduced in the present setting: on the one hand a representation of the climatology the addressee is exposed to, on the other hand the frequency of the disasters he is confronted to. Both are expressed in terms of one weather parameter, e.g. temperature or gale intensity or cumulated precipitation fallen within a period of time. The time interval, which can last for a few hours, a day, a week, will be referred to as Δ.

The frequency of the disasters might be expressed as for example in terms of the relative increase of medical emergencies occurring during a heat wave or in terms of the frequency of interventions of emergency crews in the case of a storm, all of them being expected to occur during Δ. The span of the meteorological parameter has to be specified, for example between 0 and 200 km/h for gales (at least in Switzerland), or between 0 and 400 mm precipitation, within Δ. Those inferior and superior bounds are referred as *0* and B in the sequel, with the meteorological parameter being expressed in arbitrary ***units***. Let we now specify these elements and consider Figure 1. We define:

The climate profile: upper panel, is the probability of occurrence during the period Δ of a weather event W, the intensity of which lies between q and $q + dq$:

$$C_{(q)}dq = \Pr[W \text{ occurs during } [q, q + dq]] \ ; \ q \in [0, B]$$

The exposure profile: lower panel, is the conditional probability of occurrence of a disaster D induced by a weather event W the intensity of which lies between q and $q + dq$:

$$E_{(q)}dq = \Pr[D \text{ occurs during } [q, q + dq] \mid W \text{ occurs during } [q, q + dq]]$$

Although the representation of the climate profile as the $C_{(q)}$ probability distribution function sketched above might happen to be plausible, the definition of the exposure $E_{(q)}$ yet lacks any substance and is by no way convincing. As mentioned earlier, an aim of this work consist in filling this lacuna. A glance at the title picture or at its corresponding Figure 18 indicates to which extend this attempt will be satisfied.

Both distributions are presented again in Figure 2, this time with a meteorological ***intensity threshold*** Q sketched as the downward pointing vertical arrow. One notices that most meteorological events occur at low intensity and are figured by the belly of the C distribution. Extreme events are represented by the right tail of the climatic distribution.

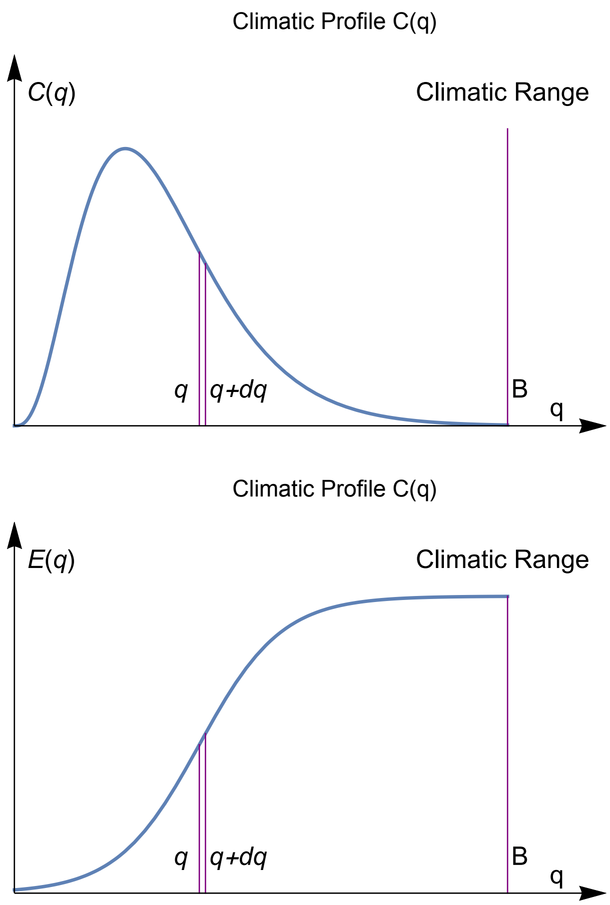


Figure 1: *Upper panel: climatic profile. Abscissa: intensity in meteorological units. Ordinate: probabilistic distribution of weather events. Lower panel: exposure profile. Abscissa: intensity in meteorological units. Ordinate: probabilistic distribution of weather induced disasters.*

4.1 Probabilistic Hit Rate and False Alarm Rate

We now extend the basic definitions provided in Section 3. Making use of the probabilistic distributions of the climatology $C_{(q)}$ and the frequency of

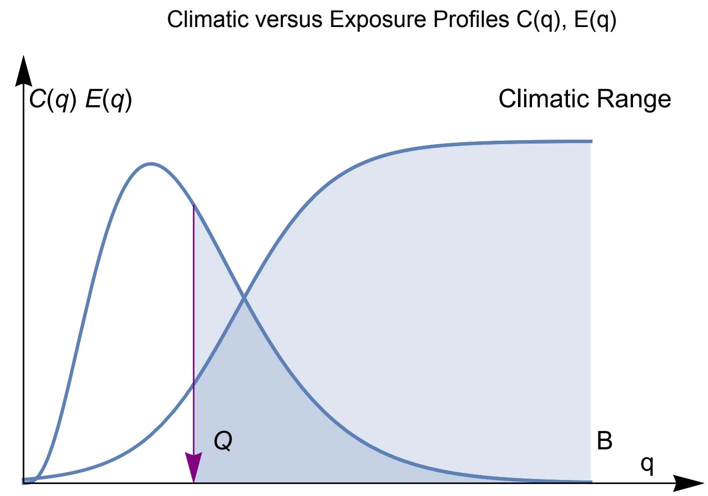


Figure 2: *Climate versus exposure. Basis for the construction of the relative operating characteristic. Abscissa: intensity in meteorological units. Ordinate: probabilistic distribution of weather events and induced disasters.*

occurrence of disasters $E_{(q)}$, we construct the following contingency Table 3 that formally corresponds to the Tables 1 and 2 introduced in Section 3. Two preliminary remarks are in order: 1) The term "case" used in the discussion below refers to the lapse of time Δ defined earlier. 2) For a given case whose weather condition is Q, the probability of occurrence of a disaster is given by $E_{(Q)}$. The probability for the weather state to be equal to Q being determined by the climatic distribution $C_{(Q)}$, the conditional probability of occurrence of a disaster triggered by an event at value Q is $E_{(Q)}C_{(Q)}$. Then, the integral $\int_{Q_1}^{Q_2} E_{(q)}C_{(q)}dq$ measures the conditional probability to experience a disaster triggered by an meteorological event whose intensity is comprised between two meteorological bounds $Q_1 < Q_2$.

Now, our probabilistic contingency takes the following shape:

		Event		Table 3
Range	Warning	did not occur	did occur	Sum
$[Q, B]$	issued	$\int_Q^B (1 - E_{(q)}) C_{(q)} dq$	$\int_Q^B E_{(q)} C_{(q)} dq$	$\int_Q^B C_{(q)} dq$
$[0, Q)$	not issued	$\int_0^Q (1 - E_{(q)}) C_{(q)} dq$	$\int_0^Q E_{(q)} C_{(q)} dq$	$\int_0^Q C_{(q)} dq$
$[0, B]$	Sum	$1 - \Omega$	Ω	1

The upper row of the table, labelled $[Q, B]$, represents the cases at and beyond the threshold Q, for which warnings are issued and mitigating actions are undertaken. The middle row of the table, labelled $[0, Q)$, represents those cases where the value of the meteorological parameter lies beneath the threshold and no warnings are issued. The lower row of the table, labelled $[0, B]$, spans all the cases where warnings are, respectively are not issued.
Let we now chase in the table and consider first the entry $\{[Q, B]$, event did occur$\}$ of value $\int_Q^B E_{(q)} C_{(q)} dq$. This integral between Q and B measures the probability of occurrence of disasters within this domain, where warning are issued. Below, the integral $\int_0^Q E_{(q)} C_{(q)} dq$ measures the probability of occurrence of disasters within the complementary domain $[0.Q]$ in which no warnings are issued. The sum of both, $\Omega = \int_0^B E_{(q)} C_{(q)} dq < 1$, measures the probability of occurrence of weather induced disasters.
Considering now the entry $\{[Q, B]$, event did nor occur$\}$ of value $\int_Q^B (1 - E_{(q)}) C_{(q)} dq$, one notices that it can be written $\int_Q^B C_{(q)} dq - \int_Q^B E_{(q)} C_{(q)} dq$. The first integral in the entry $\{[Q, B]$, sum$\}$ represents the probability of occurrence of weather within the corresponding domain, disregarding any disastrous consequence. The difference between both integrals therefore measures the frequency of the cases where warnings are issued although no disasters occur. Accordingly, the entry $\{[0, Q]$, event did not occur$\}$ with $\int_0^Q (1 - E_{(q)}) C_{(q)} dq$ measures the frequency of the cases where neither were warnings issued nor did an event occur.
The bottom row $[0, B]$ gives for both cases where disasters occur, or not occur, the corresponding sums expressed in terms of Ω. The bottom right corner is the sum on the last column as well as on the last row. It expresses the standard propriety of a probabilistic distribution: $\int_0^B C_{(q)} dq = 1$.

Conclusively, Tables 1 and 3 having the same structure, according to their definitions given in Section 3, the two following ratios are directly read on Table 3. They are:

Climatic Burden expresses the impact of the climate on the addressee's business.

$$\Omega = \int_0^B E_{(q)} C_{(q)} dq. \tag{1}$$

Hit Rate expresses the ratio between the probability of disasters occurring in the domain $[Q, B]$, for which mitigating actions are undertaken and the overall probability of occurrence of weather induced disasters. Following the definitions given in Section 3 and taking into account the correspondence between Tables 1 and 3, $\int_Q^B E_{(q)} C_{(q)} dq$ is substituted for d and Ω is substituted for $b + d$, thus giving:

$$H_{(Q)} = \frac{1}{\Omega} \int_Q^B E_{(q)} C_{(q)} dq. \tag{2}$$

False Alarm Rate expresses the ratio between the probability of non occurrence of disasters in the domain $[Q, B]$ for which mitigating actions are inadequately undertaken and the overall probability of occurrence of weather conditions not triggering disasters. Following the definitions given in Section 3 and taking into account the correspondence between Tables 1 and 3, $\int_Q^B (1 - E_{(q)}) C_{(q)} dq$ is substituted for c and $1 - \Omega$ is substituted for $a + c$, thus giving:

$$F_{(Q)} = \frac{1}{1 - \Omega} \int_Q^B (1 - E_{(q)}) C_{(q)} dq. \tag{3}$$

Three remarks are in order:

1) The variable of these functions, the meteorological threshold Q, is the lower integration bound on the right hand side of each expression.

2) The integration occurs in both cases from Q up to the upper limit B, set at a value such that the climatic probability distribution function vanishes: $C_{(B)} \simeq 0$. Not only events occurring at intensity threshold Q are taken into consideration, but all events potentially occurring beyond this threshold[1].

3) As $F_{(Q)}$ is required to belong to the interval $[0, 1]$, the exposure must satisfy $E_{(q)} < 1$. This requirement will be verified in the sequel.

[1] Financiers say: "Fat tails matter."

4.2 Relative Operation Characteristic

The Relative Operation Characteristic (ROC) is a parametrized curve expressed in $\{H_{(Q)} \times F_{(Q)}\}$ coordinates, as defined in equations (2) and (3), and represented on the domain $\{[0,1] \times [0,1]\}$, Figure 3. The parameter Q is the lower integration bound in expressions (2) and (3), ranging between 0 and B.

Taking the characteristic shape drawn below, with both extremities anchored at $(1,1)$ for $Q = 0$ and $(0,0)$ for $Q = B$, the ROC[2] is solely used in this work for the derivation of a first expression of the addressee's exposure $E_{(Q)}$. To this purpose, a simple "economic profile" of the addressee is established first, sketched as the green straight line tangent to the ROC and the array of parallels on Figure 3.

5 Economic function in the $\{H, F\}$ frame

The addressee's economic profile is expressed as a simple function, following a generalization of the theory proposed by Richardson, [2 and 3]. It provides a measure of the economic impact induced by the occurrence of four possible situations described in the following contingency Table 4. The elements are expressed in arbitrary monetary units: L represents the loss induced by a disaster for which no mitigating actions were taken. C represents the costs induced by such mitigating actions. They are due in case of occurrence of a correctly warned event, as well as in case of a mistakenly warned non event. Furthermore, mitigating actions almost never enable a total protection against disasters, accordingly, the λ factor is introduced, representing the residual costs remaining in the case of a correctly warned event. It is assumed that $\lambda \ll L$ and $C < L$.

	Event	Table 4
Mitigating actions	did not occur	did occur
taken	C	$C + \lambda$
not taken	0	L

The average cost M the addressee is faced to during a period long enough

[2]Formally, ROC is usually defined as the area located below the curve. However, the curve is frequently dubbed "ROC" for itself.

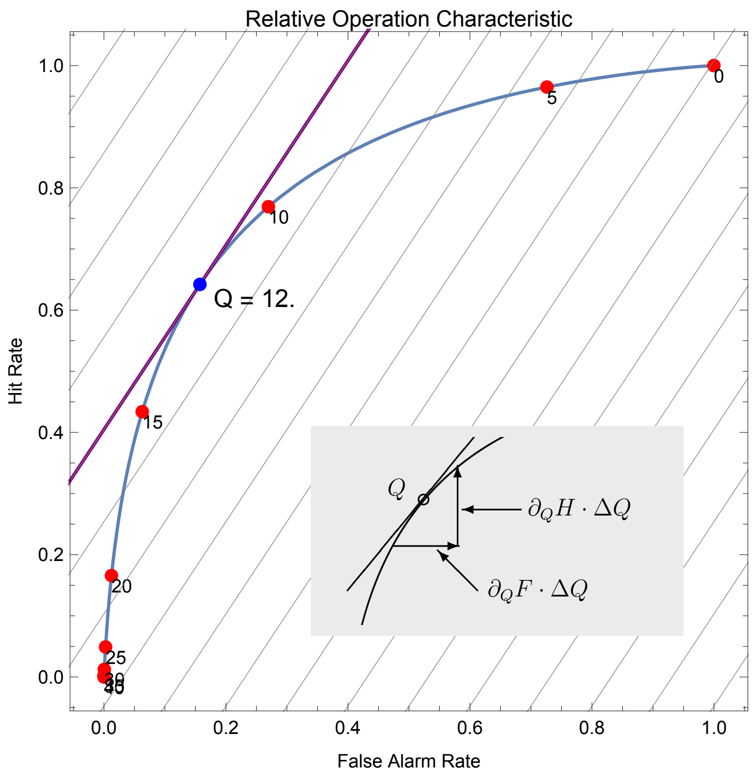


Figure 3: *Relative Operating Characteristic computed according to expressions (2) and (3). Abscissa: false alarm rate; ordinate: hit rate. Curvilinear parametrization: intensity threshold Q. The array of straight lines are iso-costs lines of the economic function. Their slope is determined by the cost-loss and residual loss ratios introduced in the text. Thanks to the concavity of the ROC, only one iso-cost is tangent to the ROC, in this case at* $Q = 12$ *units. The inbox diagram expresses the geometrical condition formulated in equation (6).*

to be of climatological relevance can now be evaluated. It is given by:

$$M = \frac{1}{a+b+c+d}\,[bL + Cc + (C+\lambda)d] \tag{4}$$

with a, b, c, d, H and Far as defined in Table 1. This expression can be formulated in terms of hit rate H and false alarm rate F, and happens to become (the derivation is presented in Annex 14.1.2.):

$$M_{R(H_{(Q)},F_{(Q)})} = L\,[\Gamma\;F_{(Q)}\;(1-\Omega) + H_{(Q)}\;\Omega\;(\Gamma+\Lambda-1) + \Omega]. \tag{5}$$

The subscript "R" in $M_{R(H_{(Q)},F_{(Q)})}$ refers to "Richardson". Two important parameters appear in the derivation. Being ratios of putatively financial terms, they are dimensionless:

Cost-loss ratio $\Gamma = \frac{C}{L}$

Residual-loss ratio $\Lambda = \frac{\lambda}{L}$

Both ratios are functions of the meteorological intensity q: $\Gamma_{(q)}$ and $\Lambda_{(q)}$, and are expected to increase under severe weather conditions. An example is pictured in the inbox of Figure 4. This dependency, that will become highly relevant in Section 11, has no influence of the following derivation.

6 Addressee's profile

Being expressed in hit rate (H) and false alarm rate (F), the economic function $M_{R(H,F)}$ can be projected onto the ROC-frame $\{[0,1]\times[0,1]\}$, as depicted in Figure 3. As it is linear in H and in F, its isolines, computed for a constant monetary values $M_R(H,F) = constant$, are straight lines, called ***iso-costs***. Moving along an iso-cost results in no change in the financial burden the addressee is faced to, and the costs are minimum at high hit rate and low false alarm ratio, in the upper left corner of the diagram. Thanks to its concavity, there only one iso-cost tangent to the ROC to which minimal costs correspond, as sketched in Figure 3. Isolines located beyond this tangency point are not reachable according to the hit & false alarm rates of the warning system. Isolines crossing the ROC are financially sub-optimal. Thus, considering the inbox diagram in Figure 3, one notices that both sides of the following expression have to be equated:

$$\frac{\partial_Q H}{\partial_Q F}|_{ROC} = \frac{\partial H}{\partial F}|_{Eco} \tag{6}$$

Let we now evaluate both sides of this equation.

6.1 ROC: $\frac{\partial_Q H}{\partial_Q F}|_{ROC}$

The propriety of the derivative of an integral with respect to its integration bound Q : $\partial_Q \int_Q^B f(q)dq = -f_{(Q)}$, is used here, yielding:

$$\begin{aligned}
\frac{\partial_Q H}{\partial_Q F}|_{ROC} &= \frac{\partial_Q[\frac{1}{\Omega}\int_Q^B E_{(q)}C_{(q)}dq\ ]}{\partial_Q[\frac{1}{1-\Omega}\int_Q^B (1-E_{(q)})C_{(q)}dq\ ]} \\
&= \frac{\frac{1}{\Omega}E_{(Q)}C_{(Q)}}{\frac{1}{1-\Omega}(1-E_{(Q)})C_{(Q)}} \\
&= \frac{E_{(Q)}}{1-E_{(Q)}} \cdot \frac{1-\Omega}{\Omega}.
\end{aligned}$$

.

$E_{(Q)} < 1\ \forall Q \in [0,B]$ has to be satisfied and $\Omega \in [0,1)$. In such circumstances, $\frac{\partial_Q H}{\partial_Q F}|_{Risk} > 0$. Furthermore, $\partial_Q(\frac{\partial_Q H}{\partial_Q F}|_{Risk}) = \frac{E'_{(Q)}}{(1-E_{(Q)})^2} \cdot \frac{1-\Omega}{\Omega} > 0$ for $E'_{(Q)} > 0$. These two requirements, $E_{(Q)} \in [0,1)$ and $E'_{(Q)} > 0$, assure the concavity of the ROC and therefore, together with the linearity of the iso-costs, the unicity of the solution of the optimisation problem.

6.2 Economic: $\frac{\partial H}{\partial F}|_{Eco}$

Taking into account the continuity of the economic function $M_{R(H,F)}$ with respect to its variables H and F, the theorem of the implicit function (Annex 14.2) is applied in the sense that the iso-costs of the Richardson function are its implicit functions. On any iso-cost of monetary value $M_\$$ the relation $M_R(H,F) = M_\$$ holds for the corresponding H and F values. Thus, if H is the implicit function expressed in terms of F, the total derivative is:

$$D_F M_{R(H_{(F)},F)} = \frac{\partial M_R}{\partial H}\frac{\partial H}{\partial F} + \frac{\partial M_R}{\partial F} = 0.$$

Therefore, after few algebraic manipulations, the result reads (with "*Eco*" standing for "*Economic*"):

$$\begin{aligned}
\frac{\partial H}{\partial F}|_{Eco} &= -\frac{\partial M_R}{\partial F}(\frac{\partial M_R}{\partial H})^{-1} \\
&= \Gamma L \frac{\Omega - 1}{\Omega(\lambda + L(\Gamma - 1))} \\
&= \frac{1-\Omega}{\Omega} \cdot \frac{\Gamma}{1 - \Lambda - \Gamma}.
\end{aligned}$$

6.3 Synthesis:

Requiring the identity of the slopes at the point on the ROC with curvilinear abscissa Q and equating the two quantities according to Equation (6) $\frac{\partial_Q H}{\partial_Q F}|_{ROC} = \frac{\partial H}{\partial F}|_{Eco}$ leads to:

$$\frac{E_{(Q)}}{1 - E_{(Q)}} \cdot \frac{1-\Omega}{\Omega} = \frac{1-\Omega}{\Omega} \cdot \frac{\Gamma}{1 - \Lambda - \Gamma}$$

$$E_{(Q)} = \frac{\Gamma_{(Q)}}{1 - \Lambda_{(Q)}} \tag{7}$$

An encouraging harbinger is the fact that the climate burden Ω has vanished: the addressees's exposure is only defined in terms relevant to him, the cost & residual loss ratios, now expressed as functions of Q.

This result, although promising, is not plainly satisfactory: $E_{(Q)}$ is likely to become greater than one as $\Lambda \longrightarrow 1$ or if Γ increases substantially. But, $E_{(Q)} > 1$ means in both cases that the mitigating costs are higher than the disaster costs: no rational addressee would trigger mitigating actions more expensive than the disaster supposed to be covered! He would rather consider the services provided by an insurance company. This is a practical reason for us to disregard exposures greater than one. Certainty about this issue will be brought in Section 10.

A satisfactory point is the fact that one easily frees oneself from financial considerations: all parameters introduced so far are dimensionless. As for example, *"If I consider that the consequences of such a calamity are ten times worser for me then the provisions I may take to avoid them, then my cost loss ratio is* 1/10. *If the residual damage I experience after having* *successfully triggered my mitigating actions is still one fifth of the damage induced by the calamity, my exposure is* $1/8 = 12.5\%$*".*

Finally, as promised, cost loss and residual loss ratios may be expressed as functions of the weather intensity Q. An example of such a dependency is provided in the inbox in Figure 4, where the exposure increases stepwise in response to the raising weather intensity. Obviously, only informed addressees well aware of the vulnerable components of their businesses might be able to produce such accurate patterns.

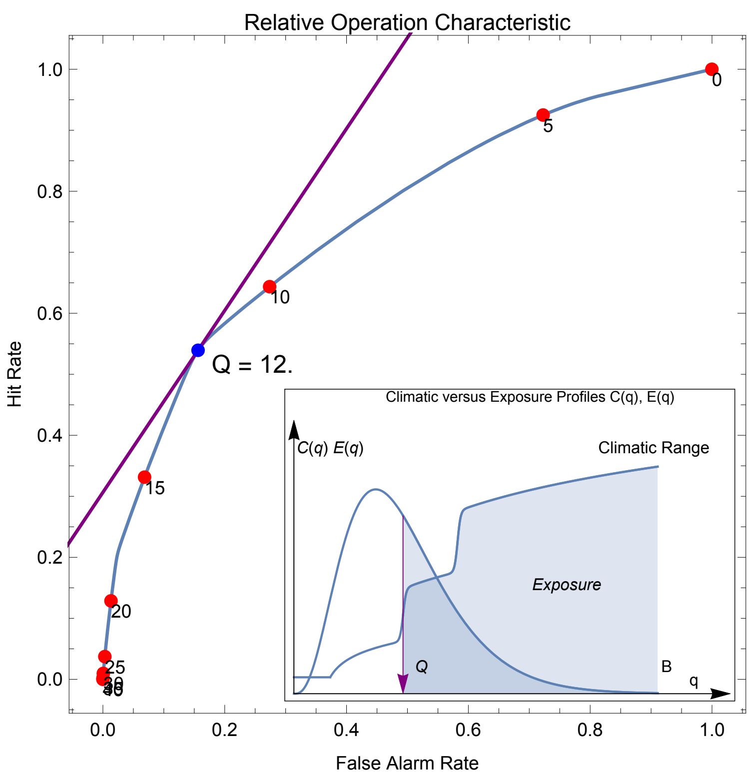


Figure 4: *ROC modified as the exposure becomes dependent on the stepwise increasing cost loss ratio with respect of the weather intensity (inbox). The kink on the ROC at* $Q = 12$ *occurs at the first step of the exposure.*

The next Sections will be devoted to the construction of a new definition of the exposure, this time superiorly bounded by one. To this purpose, a profile of the warning issuer will be established first, then a new definition of the hit rate and false alarm ratio will be introduced, taking into account probabilistic forecasts delivered by ensemble forecasting systems. The economic function will then be reframed, finally allowing us to reformulate the exposure within this new context.

7 Issuer's profile

Having discussed the addressee's characteristics so far, we will now focus our attention on the issuer, with meteorological services in mind. Considering that meteorological warnings are increasingly based on probabilistic forecasts, possibly emanating from ensemble prediction systems, a short description of a probabilistic decision scheme is provided first.

7.1 Probabilistic warning decision

Characteristics of the simulated probabilistic forecasts and the corresponding weather events are presented in Figure 5. The climatic range in which forecasted weather events occur is represented by the abscissa, spanning from 0 to an upper bound B arbitrarily set at $B = 30$ in the Figure. The probability of their occurrence is expressed on the ordinate. The downward pointing black arrow determines a meteorological threshold Q. A probabilistic forecast for the next verifying time is sketched as the gaussian probabilistic distribution function[3]. The area below the curve $f(q)$ located at the right of the meteorological threshold Q, given by $\displaystyle\int_Q^B f(q)dq$, expresses the probability of the event "Weather occurs with an intensity equal or greater than Q", as forecasted by the simulated ensemble prediction system. (with $\int_0^B f(q)dq = 1$). Having these elements at hand, it suffices now to introduce besides the meteorological threshold Q a ***probability threshold*** P to define a simulated warning system.

The choice of a decision scheme is already a decision *per se*, indeed a "*meta-decision*". In this perspective, seeking rationality, the system applied in the sequel is a simple automatic algorithm operating in accordance to the rules given in Figure 5 and Table 5 below.

[3]Of course, in the practice, the distribution is almost never gaussian

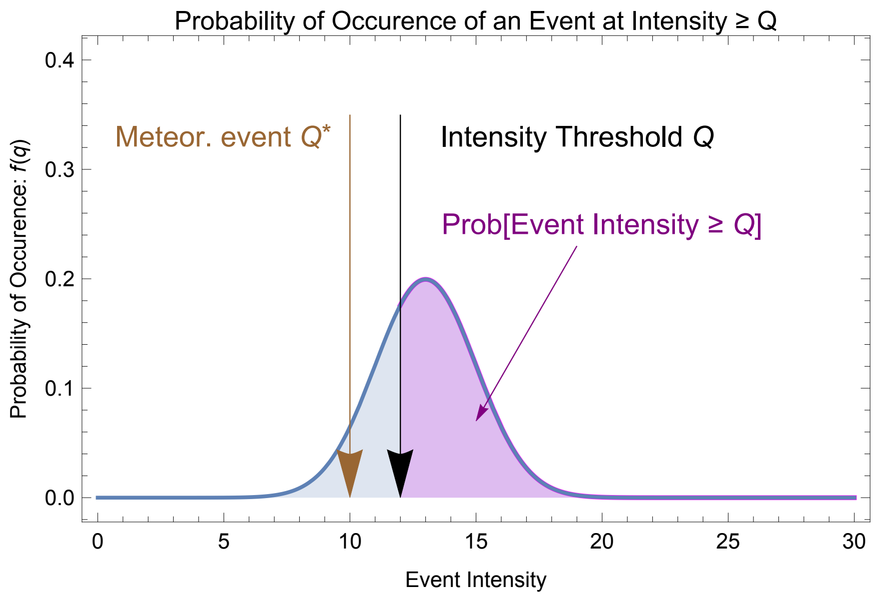


Figure 5: *Decision based on a probabilistic forecast. Q is the meteorological threshold. The area below the probability distribution function $f(q)$ located at the right of the (meteorological) intensity threshold Q expresses the probability of the event "Weather occurs with an intensity equal or greater than Q", as forecasted by the ensemble prediction system.*

Threshold		$Q^* < Q$	$Q^* \geq Q$	Table 5
	Event	did not occur	did occur	
Probability	Warning			
$\int_Q^B f(q)dq \geq P$	issued	False alarm	Hit	
$\int_Q^B f(q)dq < P$	not issued	Correct rejection	Miss	

Reflecting on the interplay between the probability and the intensity thresholds, the reader have noticed that the latter is determined by the addressee. He may suspect that the former will be the decisional tool in issuer's hands. A new formulation of the hit rate and the false alarm ratio will clarify the issue.

7.2 Probabilistic Hit Rate and False Alarm Ratio

We now evaluate the warning system according to the probabilistic setting just defined. In this representation, a distinction is made between cases where an event occurred, $Q^* \geq Q$, and those where no event occurred, $Q^* < Q$, following Table 5.

It is reasonable to expect from a reliable warning systems that it delivers mostly high probabilities when events actually arise and rather low probabilities when they do not. This expectation is pictured in Figure 6: the curves $u_{(p)}$ and $v_{(p)}$ depict the distributions of probabilities delivered by a fairly reliable warning system. Events $Q^* \geq Q$ are collected at rather high probabilities, as expressed by the $u_{(p)}$ distribution. Non events $Q^* < Q$ are mostly collected at low probabilities, following the $v_{(p)}$ distribution. Both distributions are referred to as $\{u_{(p)}, v_{(p)}\}|_Q$. This procedure collects the probabilities delivered by the forecasting system as a rank histogram or a "Talagrand Diagram" would do. The difference here consist in dividing the rank histogram into two the sub-classes: "event did -" and "event did not occur". The area located below curve $v_{(p)}$ in the domain $p \in [0, P)$ corresponds to the entry "a" in the Table 1. The area located below curve $u_{(p)}$ in the domain $p \in [0, P)$ corresponds to the entry "b" in the same table. The area located below curve $v_{(p)}$ in the domain $p \in [P, 1]$ corresponds to the entry "c" in the table. The area located below curve $u_{(p)}$ in the domain $p \in [P, 1]$ corresponds to the entry "d" in the table. Introducing the weighting factor $\Pi = \int_0^1 u_{(\pi)} d\pi$, following definitions can be stated in this new context:

Hit Rate at probability threshold p: quotient of the area located below the $u_{(p)}$ curve integrated between p and 1 and the total area under the curve, integrated between 0 and 1:

$$H_{(p)} = \frac{1}{\Pi} \int_p^1 u_{(\pi)} d\pi \tag{8}$$

False Alarm Ratio at probability threshold p: quotient of the area located below the $v_{(p)}$ curve integrated between p and 1, and the total area under both $u_{(p)}$ and $v_{(p)}$ curves, integrated between p and 1:

$$Far_{(p)} = \frac{\int_p^1 v_{(\pi)} d\pi}{\int_p^1 (u_{(\pi)} + v_{(\pi)}) d\pi} \tag{9}$$

Frequency Bias at probability threshold p computed according to its definition $FB_{(p)} = \frac{H_{(p)}}{1-Far_{(p)}}$:

$$FB_{(p)} = \frac{1}{\Pi} \int_p^1 (u_{(\pi)} + v_{(\pi)}) d\pi \tag{10}$$

All this reflection is related to the intensity threshold Q. Accordingly, the two functions are referred to as $\{H_{(p)}, Far_{(p)}\}|_Q$. This requirement will be treated and extended in Section 11.

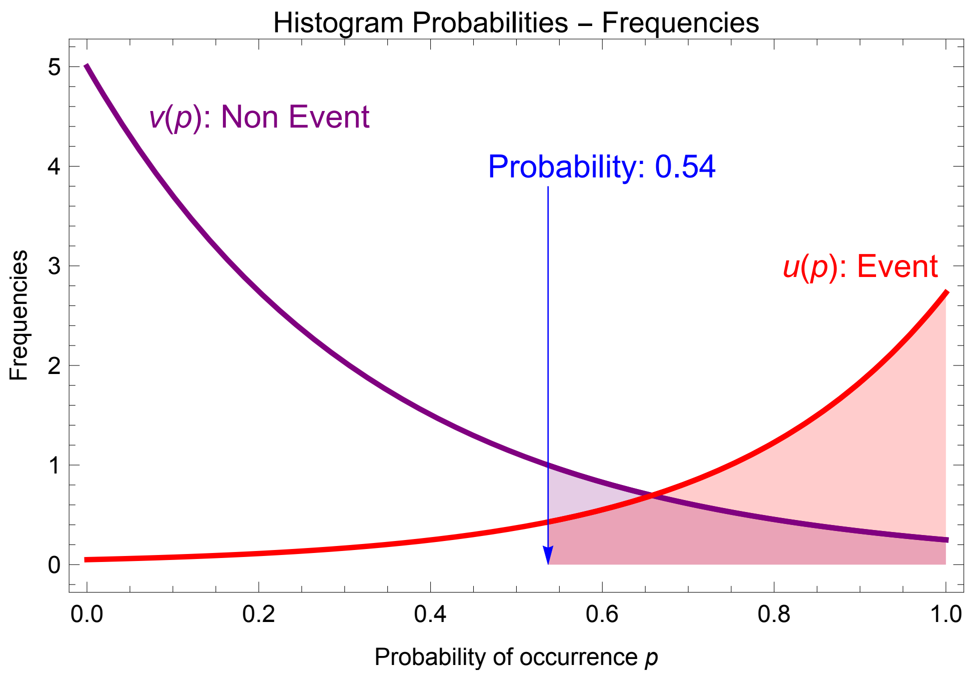


Figure 6: *Distribution of warning probabilities delivered by an ensemble forecasting system tuned at intensity threshold Q. Abscissa: probability at which warnings are issued at probability threshold p, depicted by the downward pointing vertical arrow. Ordinate: frequency of occurrences of probabilities, expressed in case of occurring, respectively not occurring events.*

We are now in the position to build a new ROC laid in the hit rate - false alarm ratio frame, presented in following Figure 7. This time, however, the ROC is parametrized in terms of probabilities. The lines radiating from the lower right corner are Iso Frequency-Bias Lines. Expressed as $FB = \frac{H}{1-Far} = Cste$, they build a new, intrinsic reference grid within the $\{H, Far\}$ frame without reference to any specific warning system.

Figure 7 shows us that if warnings are issued at low probability thresholds, located on the right upper branch of the the ROC, the addressee will statistically receive more warnings than the number of event he may expect to be confronted to. In this sense, with a frequency bias greater than one, he will be "over-warned". On the contrary, the addressee will be "under-warned" if higher probability thresholds are favored, then located on the lower left part of the ROC. Accordingly, the frequency bias will be lower than one.

A less intuitive notion ***risk awareness*** can now be introduced, depicted in Figure 7. Addressees who prefer to receive more warnings than the number of weather events they are confronted to are ***risk adverse***. They will receive warnings delivered at a frequency bias $FB > 1$ and at low probability thresholds. *Mutatis mutandis*, at frequency bias $FB < 1$, addressees will be capable to cope effortlessly with severe weather events and be qualified as ***risk friendly***. They will then be satisfied with seldom warnings delivered at higher probability thresholds. On notices on the figure that a probability threshold 1/2 does not necessarily correspond to a frequency bias of one. This is the reason why the concept of risk awareness is expressed in terms of frequency bias instead of being related to probabilities of occurrence.

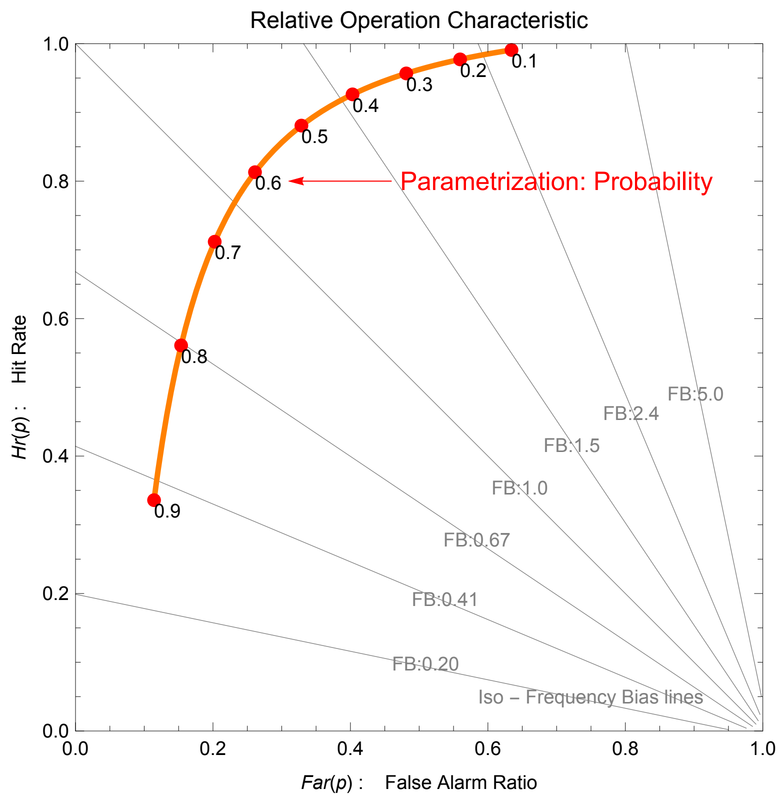


Figure 7: *Relative Operation Characteristic in the hit rate - false alarm ratio frame, computed according to expressions (8) and (9) at intensity threshold Q. Abscissa: false alarm ratio. Ordinate: hit rate. Curvilinear parametrization: probability threshold. The straight lines radiating from the bottom right corner are iso frequency-bias lines. Low probability thresholds correspond to frequency bias greater than one, and conversely. The frequency bias is neutral - equals one - on the diagonal. One notices that a neutral frequency bias is reached at a probability threshold different from 1/2.*

7.3 Interlude: trade-off between risk and reward

The shape of the ROC appears in various economic settings and is characteristic for processes described by two quantities connected in the sense that the increase of one of them yields the decline of the other, and reciprocally. Such interactions are mostly informative when the quantities are of different nature. On the contrary, settings where plus of something for one player induces less of the same thing for another player correspond to null-sum outcomes in the game theory.

In our situation, however, the trade-off occurs between hit rates and false alarm rates or ratios and enables the definition of new concepts such as frequency bias or exposures. Two further examples of this trade-off are presented in Figure 8. The former on the left panel is well known and belongs to the realm of finances, the latter on the right panel to the field of economic policy.

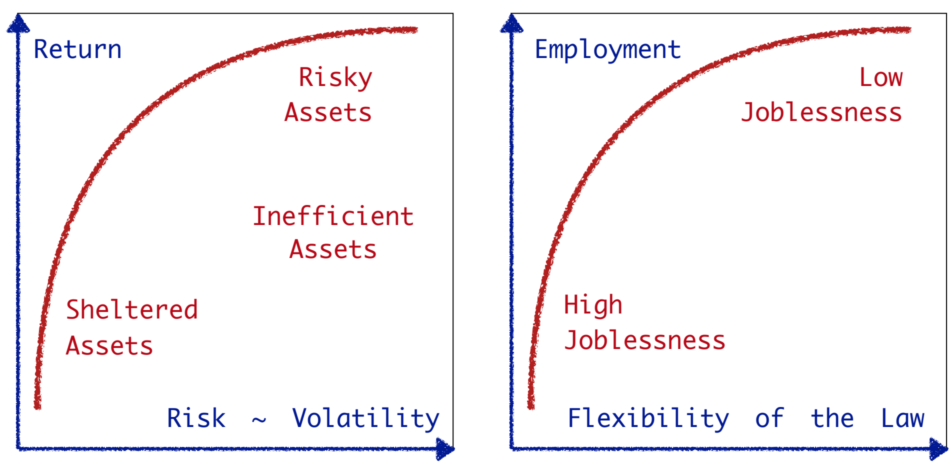


Figure 8: *Left Panel: efficient frontier on a financial market. Right Panel: empirical relationship between the employment rate and the flexibility of the work law in an economy.*

The relationship shown on the left panel was formalized by Marcowitz in the sixties of the previous century under the name of "capital asset pricing model". It claims and describes how a risky investment is likely to yield higher gains than an innocuous one, at the price of a higher risk. Risk adverse investors prefer sheltered assets at the price of low revenues. Inefficient assets are at the same time risky and deliver low returns. In this financial context, the relative operation characteristic is named the "efficient frontier". The right panel shows an empirical attempt to describe the relationship between the unemployment rate occurring in an economy and the impact of the legal thrall on the working rules applied in this economy.

8 Optimal decision, first approach

The addressee's economic profile has now to be rewritten in terms of hit rate and false alarm ratio in order to be expressed in this new frame. This operation is described in the following SubSection.

8.1 Economic function in the $\{H, Far\}$ frame

The elements presented in Section 5, Table 4, remain valid and are repeated here: L represents the loss induced by a disaster for which neither a warning was issued nor mitigating measures were taken. C represents the costs induced by mitigating measures, due in case of occurrence of a correctly warned event, as well as in case of a mistakenly warned non-event. λ represents residual disaster costs remaining in the case of a correctly warned event.

The average costs the addressee is faced to during a period long enough to be of climatological relevance, expression (4), are:

$$M = \frac{1}{a+b+c+d}\left[bL + Cc + (C+\lambda)d\right] \tag{11}$$

with a, b, c, d, H and Far as defined in Table 1 and computed following the decision scheme presented in Figure 5 and Table 5, Section 7.

The derivation is presented in Appendix 14.1.1. The resulting function, hereafter referred to as M_A, reads:

$$\begin{aligned} M_{A(H,Far)} &= L\,\Omega\,[(1-H) + H\,\Lambda + \Gamma\,\frac{H}{1-Far}] & (12)\\ &= L\,\Omega \cdot \begin{bmatrix}1 & \Lambda & \Gamma\end{bmatrix} \cdot \begin{bmatrix} 1-H \\ H \\ H\,(1-Far)^{-1}\end{bmatrix}. & (13)\end{aligned}$$

It can be written either as a standard algebraic expression, or as a scalar product, as in expression (13). The significance of this expression appears by introducing the following terms:

Climatic Burden: $L\,\Omega$

Addressee's profile: $[1,\, \Lambda,\, \Gamma]$

Issuer's Profile: $[1 - Hr,\, Hr,\, \frac{H}{1-Far}] \;=\; [1 - Hr,\, Hr,\, FB]$

The issuer's Profile can be expressed either in terms of hit rate and false alarm ratio, as usual, or, newly, as a function of the hit rate and the frequency bias. Summarizing in words instead of symbols and expressing the scalar product in brackets familiar to quantum physicists, the economic function now reads:

Economic Function $=$

Climatic Burden $\cdot <$ ***Addresse's Profile*** $|$ ***Issuer's Profile*** $>$

Three eminent characters of our plot, now nicely staged in an easily grasped linear expression, tell us that the approach based on the false alarm ratio happens to be richer than the corresponding approach making use of false alarm rate. This feat will gain full limpidity in the next Section.

Profiles of both the issuer - the ROC -, and the addressee - his economic profile - are now expressed in terms of hit rate and false alarm ratio. They can be superposed as an $\{H_{(P)} \times Far_{(P)}\}$ curve, respectively as iso-costs lines of the $M_{A(H,Far)}$ function onto the $\{[0,1] \times [0,1]\}$ square, as presented in Figure 9. A great deal of information being available on this Figure, our task will now consist in deciphering everything.

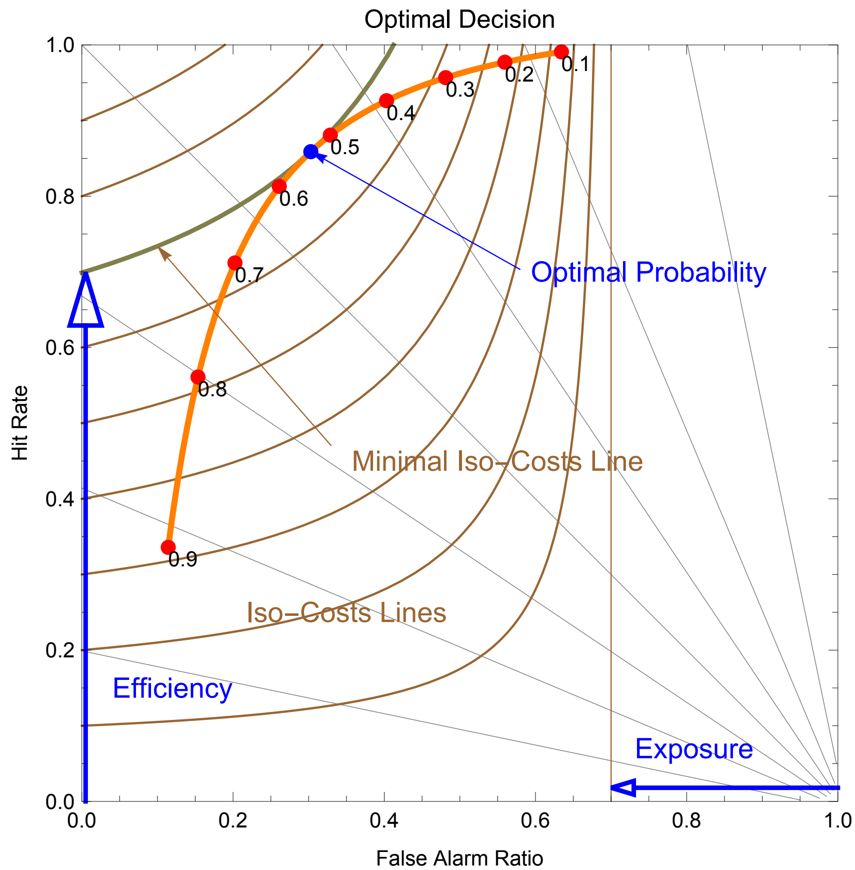


Figure 9: *The relative operation characteristic in the* $\{H_{(p)}, Far_{(p)}\}|_Q$ *at intensity threshold* Q *is drawn in orange. The iso-costs of the economic function* $M_{A(H,Far)}$ *build the network of brown curves (that are indeed hyperboles). The efficiency of the warning system and the addressee's exposure are pictured as blue arrows. The optimal probability pointed to on the ROC represent the optimal decision sought after.*

Following the same methodology as in Section 6, it is considered that the addressee will seek out his minimum financial burden in the long term, and will therefore require to be warned at probability thresholds providing the minimum value of the economic function $M_{A(H,Far)}$. Noticing that the addressee's profile, represented by the iso costs lines, is convex and the issuer's profile - the ROC - concave, it appears that the optimal probability threshold P^* is naturally located at the point of tangency of both profiles, pointed as the blue dot in Figure 9. Accordingly, repeating the procedure applied in Section 6, the tangency requirement is expressed in this new context as:

$$\frac{\partial_P H}{\partial_P Far}|_{Issuer} = \frac{\partial H}{\partial Far}|_{Addressee} \tag{14}$$

8.2 The addressee's perspective

Taking into account the continuity of $M_{A(H,Far)}$ with respect to its variables H and Far, the theorem of the implicit function (Annex 14.2) is applied in the sense that the iso-costs of $M_{A(H,Far)}$ are its implicit functions. On any iso-cost of monetary value $M_\$$ the relation $M_{A(H,Far)} = constant$ holds for the corresponding H and Far values. Thus, if H is the implicit function expressed in terms of Far, the total derivative is:

$$D_{Far} M_{A(H_{(Far)},Far)} = \frac{\partial M_A}{\partial H}\frac{\partial H}{\partial Far} + \frac{\partial M_A}{\partial Far} = 0.$$

After few algebraic manipulations, the result reads:

$$\begin{aligned}
\frac{\partial H}{\partial Far}|_{Addressee} &= -\frac{\partial M_A}{\partial Far}(\frac{\partial M_A}{\partial H})^{-1} \\
&= -\frac{\Gamma H}{1 - Far} \cdot \frac{1}{\Gamma + (1 - Far)(\Lambda - 1)} \\
&= \frac{E H}{(Far - 1)(E + Far - 1)}.
\end{aligned}$$

This is an ordinary differential equation that can be easily integrated, eventually yielding the expression of the iso-costs in the $\{H, Far\}$ frame. Starting with:

$$\frac{\partial H}{\partial Far} = \frac{E H}{(Far - 1)(E + Far - 1)} \tag{15}$$

one has:

$$\frac{dH}{H} = E\frac{dFar}{(Far-1)(E+Far-1)}$$

yielding, after integration (Annex 14.3):

$$H_{(Far)} = C\frac{Far-1}{E+Far-1}.$$

This expression has been used to draw the isolines in Figures 9 & 10. The integration constant C is determined with the classical technique of the "variation of the constants", implemented through the Mathematica operator *FindMinimum*. The optimal probability threshold is obtained as a direct subsequent outcome. Finally, noticing that the expression $H_{(Far)}$ is asymptotic at $E = 1 - Far$ enables us to locate and draw the exposure as the vertical asymptote on the Figure, as well as the horizontal blue arrow depicting its value.

The efficiency (also called impact) is another characteristic of the warning system, measured in relation to the addresses's exposure at intensity threshold Q. It is a ratio comprised between 0, if the system is absent, or disconnected, and 1 when it is perfect, neither missing any event nor issuing any false alarm. Efficiency is introduced in next Section 9 and reads:

$$W_{(H,Far)}|_Q = \frac{M_{A(H,Far)} - M_{A(0,0)}}{M_{A(1,0)} - M_{A(0,0)}}$$

Independent from the climate both actors are submitted to, it is pictured as the vertical blue arrow in Figures 9 & 10.

Addressees with high exposure happen to be reluctant to trigger costly mitigating actions and prefer to be confronted to their climatic fate. Favoring a frequency bias lower than 1, they are risk friendly. Contrarily, addressees requiring frequent warnings delivered at low probability thresholds are risk adverse. In all cases, to high exposures correspond lower efficiencies. Addressees may either choose to reduce the cost of their mitigating actions according to the definition of the exposure $E_{(Q)} = \frac{\Gamma_{(Q)}}{1-\Lambda_{(Q)}}$, thus working on the Γ term, or intervene on the L term of their climatic burden $L\,\Omega$. These notions will be further discussed in Section 12.

Due of the reciprocal relationship between the frequency bias and the probability threshold (the former rises when the latter sinks), an unexpected

connection emerges between the risk awareness of both actors, sketched in the following example:
You are driving on Swiss winterly roads a car that, alas, is not equipped with snow tyres. Of course, you are frightened by few snowflakes whirling in the air and, consequently, you expect a lot of bulletins to be delivered for meaningless weather events ... You are risk adverse and, requiring from the forecaster frequent warnings delivered upon weak evidence for irrelevant events, your force him or her to perform in a dared, indeed risk friendly manner. Accordingly, the frequency bias soars heedlessly. On the contrary, underway in a four-wheels drive equipped with snow tyres, chains, snow boots and a snow shovel in the luggage compartiment, you love to cross alpine passes under precarious weather conditions, almost disregarding weather bulletins. Thus, Yes! you are definitely risk friendly: only solid warnings provided for putatively extreme events draw your attention. In such cases, forecasters deliver seldom, founded bulletins. Deciding upon strong evidence, they behave risk adversely.
Relevant in both examples is the fact that the drivers act rationally, in accordance to their respective state of preparedness. The following table summarizes:

Table 6

If the addressee	then the issuer
requires early warnings delivered at low probabilities, *he is risk adverse*	must warn upon weak evidence, *he is required to behave in a risk friendly manner*
prefers late warnings delivered at high probabilities, *he is risk friendly*	can warn upon strong evidence, *he may afford to be risk adverse*

Our actors are entangled in their behaviors or, more precisely, the issuer must adapt his decisional behaviour to the addressees's inclinations.

Concluding so far, it is worth mentioning that three key elements are now interwoven: the efficiency of a warning system, the probability threshold and the addressee's exposure. Following Figure 10 depicts various combinations of them. All other theoretical scaffolds used so far have been removed.

The issuer's side $\frac{\partial_P H}{\partial_P Far}|_{Issuer}$ of Equation (14) will be treated in Section 10, Optimal decision, Second Approach.

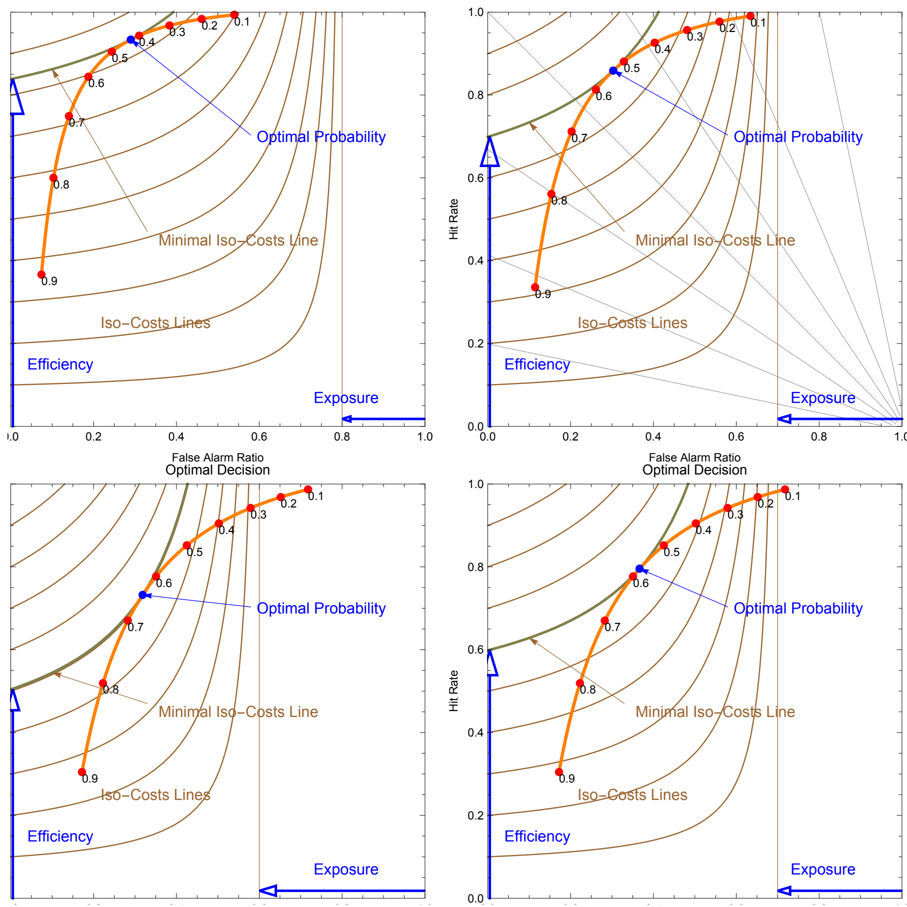


Figure 10: *Clockwise from the upper left panel: the best case: high performance, lowest exposure. Upper right panel, repetition of Figure 9 as reference. Lower right panel, same exposure as upper right, but lower performance. Lower left: the worst case, low performance and high exposure. The efficiency decreases, the probability threshold increases accordingly.*

8.3 Alternative coordinate system

We have worked so far in the cartesian coordinate system {False Alarm Ratio - Hit Rate}, which is well suited to the issuer's needs and expresses the fundamental characteristics of her warning system. However, a curvilinear coordinate system established along the {Frequency Bias - Efficiency} lines happens to be better suited to express addressee's expectations. Indeed, Frequency Bias and Efficiency are more significant to his business than to her's. Making use of the geometric properties discussed so far, both representations can be made available on a ROC diagram, as presented on Figure 11.

According to this representation, as soon as an addressee has chosen a Frequency Bias, his choice determines his Exposure, as well as the Efficiency that he can expect from the warning system. Dualistically, the issuer then knows then the probability threshold at which warnings have to be issued, as well as the corresponding Hit Rate and False Alarm Ratio.

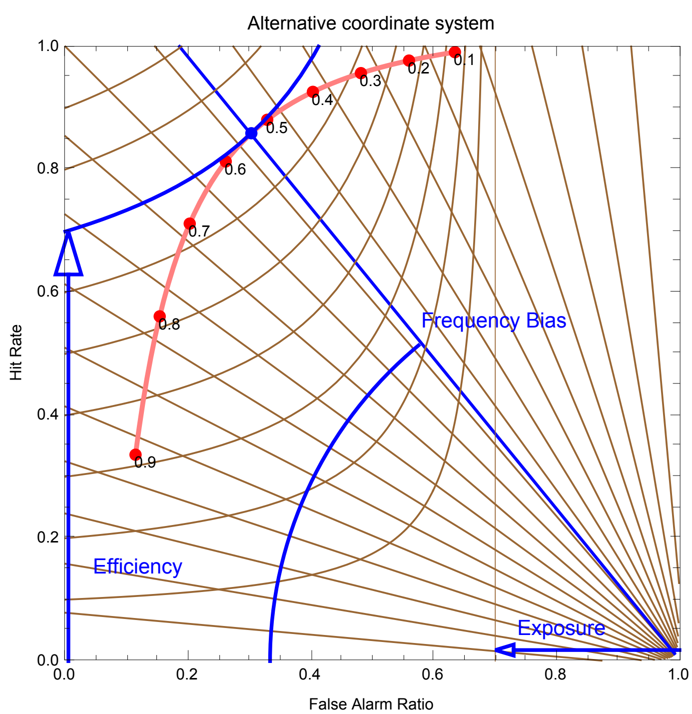


Figure 11: *Alternative curvilinear coordinate system {Frequency Bias - Efficiency}. This system expresses the addressee's expectations.*

9 Efficiency of a warning system

Few agreements have to be reached in order to derive a consensual definition of the concept of "efficiency".

A warning system is said to be ***disconnected*** if it delivers no warnings. In such a case, both the hit rate and the false alarm ratio are zero and the addressee experiences a climatic burden $M_{A(H=0,Far=0)} = L\,\Omega$, as expected. A warning system is ***perfect*** if it detects all events and issues no false alarms. Its hit rate is one, its false alarm ratio equals zero and the addressee experiences a climatic burden $M_{A(H=1,Far=0)} = L\,\Omega\,[\Lambda + \Gamma]$. Considering now an intermediate warning system operated at false alarm ratio Far and hit rate H, it can be characterized by its ***efficiency*** $W_{(H,Far)}|_Q$, defined as the following ratio[4]:

$$W_{(H,Far)}|_Q = \frac{M_{A(H,Far)} - M_{A(0,0)}}{M_{A(1,0)} - M_{A(0,0)}} \tag{16}$$

As the climatic burden $L\,\Omega$ simplifies between numerator and denominator in the fraction, the efficiency is independent form the climate the warning system and the addressee are submitted to. Of course, however, if the forecasting system alimenting a warning system experiences difficulties in specific climatic situations, these impediments have a negative impact on the efficiency: the ROC is then shifted downwards to the right, as depicted in the lower panels of Figure 10. In any cases, such an abatement in efficiency is induced by the forecasting system, not by the climate. Efficiency may be formulated in various forms, three of them deliver valuable insights:

$$\begin{aligned}
W_{(H,Far)}|_Q &= H\,\frac{1 - \frac{1}{1-Far}E_{(Q)}}{1 - E_{(Q)}} && (17)\\
&= \frac{1}{1 - E_{(Q)}}\left(H_{(p)} - FB_{(p)}\;E_{(Q)}\right) && (18)\\
&= = \frac{1}{\Pi}\left(\int_p^1 u_{(\pi)}d\pi - \frac{E_{(Q)}}{1 - E_{(Q)}}\int_p^1 v_{(\pi)}d\pi\right). && (19)
\end{aligned}$$

As $\lim_{Far\to 0} W_{(H,Far)} = H$ in the first expression, the efficiency can be directly read on the diagram in Figure 9, simply following the Iso-Cost line tangent to the ROC towards its interSection with the ordinate axe of the diagram: indeed are iso-cost lines iso efficiency lines as well. This is the

[4] "W" for german "Wirkung".

location pointed to by the vertical blue arrow in Figures 9 and 10.

The second expression provides an outstandingly simple condition for the positiveness of the efficiency: $W_{(H,Far)}|_Q > 0$:

$$FB_{(p)}\ E_{(Q)} < H_{(p)} \tag{20}$$

This is a substantial statement! It tells us that in case of weak issuer's performance (low hit rate) and high addressee's exposure (high $E_{(Q)}$), a strategy susceptible to maintain a sustainable efficiency consist in reducing the frequency bias, indeed the number of warnings issued. This is practically achieved in increasing the probability threshold. The efficiency of the warning system becomes negative if the inequality is violated. The expression is easily solved for the frequency bias and gives: $FB = (H - W(1-E))/E$. Thus, for example, if an efficiency $W \geq 1/2$ is required by a hit rate $H = 4/5$ for an addressee whose exposure is $E = 2/3$, then the frequency bias should not exceed $FB = 19/20 = 95\%$. If the frequency bias is raised from 95% to 100% without changing neither the hit rate nor the exposure, then the efficiency decreases to $W = 2/5 = 40\%$[5].

Expression (19) is provided for the sake of completeness and corresponds to the equations given in Section 7.2.

10 Optimal decision, second approach

We now consider the left hand side of equation (14). Using the definitions of the hit rate and false alarm ratio introduced in Section 7.2, equations (8) and (9) as well as equation (19), we dare to establish a direct connection between the addressee's exposure $E_{(p)}$ at intensity threshold Q, as specified in Table and Figure 5, and the distributions $\{u_{(p)}, v_{(p)}\}|_Q$ pictured in Figure 6, evaluated at the same intensity threshold.

This relation can be derived following two possible paths. The first approach makes use of the efficiency $W_{(H,Far)}|_Q$. Seeking for values of p delivering the maximal efficiency, one solves:

$$\partial_p \left[W_{(H_{(p)},Far_{(p)})} \right] = 0. \tag{21}$$

[5]This elementary estimation is done *ceteris paribus*. In reality, all parameters evolve simultaneously.

This calculation, classical, is presented in Annex 14.4.1 and yields:

$$E_{(Q)} = \frac{1}{1 + \frac{v_{(p)}}{u_{(p)}}}. \tag{22}$$

As $E_{(Q)} = \frac{\Gamma_{(Q)}}{1-\Lambda_{(Q)}}$ depends on the intensity threshold Q only, an optimal probability threshold ${p^*}_{(Q)}$ can be computed for each value of Q through the inversion of the above expression.

The second calculation is more convoluted. The exposure is extracted from Equation (15) and explicitly calculated according to all terms figuring in the right hand side:

$$E_{(p)} = \frac{(Far-1)^2}{H\frac{\partial Far}{\partial H} - Far + 1}$$

The calculation, presented in Annex 14.4.2, yields:

$$E_{(p)} = \frac{1}{1 + \frac{v_{(p)}}{u_{(p)}}}|_Q \tag{23}$$

In this perspective, the exposure depends on terms relevant to the issuer only. It is formulated as a function of the probability threshold, itself related to the intensity threshold given in Table 5 and expressed in terms of the $\{u_{(p)}, v_{(p)}\}|_Q$ functions depicted in Figure 6. Four remarks are in order:

1) no preliminary hypothesis has been made that would specify the shape of the $u_{(p)}$ and $v_{(p)}$ distributions.

2) As both $u_{(p)} > 1$ and $v_{(p)} > 1$ - they are strictly positive by construction - the exposure is bounded between 0 and 1: $0 < E_{(p)} < 1$, and the requirement expressed in Section 6.3. is fulfilled.

3) The exposure has a sigmoidal (lazily smoothed *"S"*) shape. It is exactly sigmoidal if both $u_{(p)}$ and $v_{(p)}$ are raising, respectively decreasing exponential functions of p.

4) The sharpness of the sigmoid is determined by the quality of the warning system (included all observing and forecasting systems feeding it).

10.1 The issuer's perspective

We now consider some properties of the exposure in expressed in its new formulation. Figure 12 presents the exposure (blue curve) computed according to Equation (22), expressed in terms of the probability threshold p.

The efficiency is depicted as the concave light grey curve. The $\{u_{(p)}, v_{(p)}\}|_Q$ functions (light purple and red curves in the background) are those of Figure 6. The system is tuned by the choice of a probability threshold that maximises the efficiency. An alternative consist in choosing another exposure (on the ordinate) that determines the corresponding probability threshold and implies another efficiency, not depicted here.

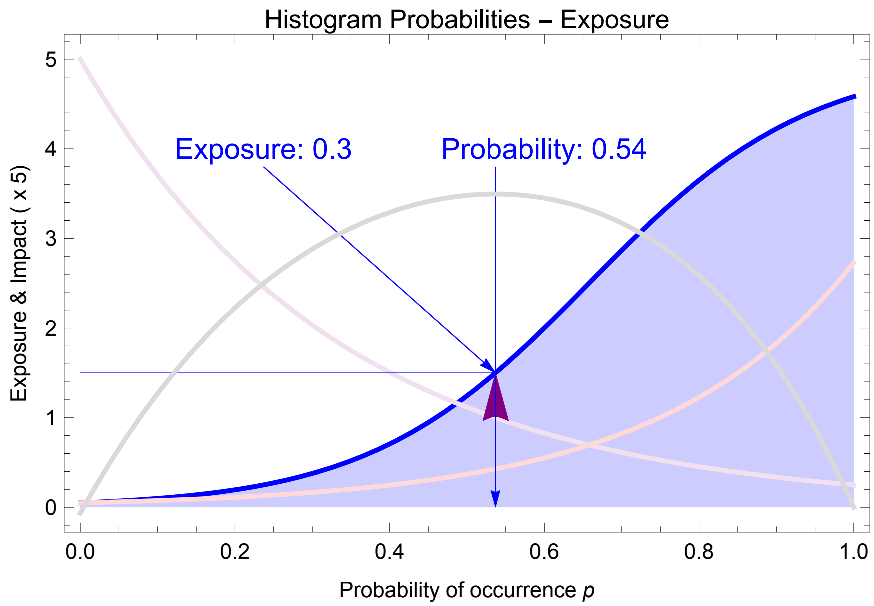


Figure 12: *Sigmoidal shape of the exposure (blue curve) computed according to Equation (22). The concave grey curve depicts the efficiency (impact) that reaches its maximum* $W = 0.7$ *at optimal probability threshold* $p = 0.54$*, thus leading to exposure* $E = 0.3$*. All figures correspond to the values pictured in Figure 9*

The diagram presented in Figure 13 is a sagittal attempt at describing the tuning sequence of a warning system. Choosing a probability threshold primarily determines the exposure, and settles the frequency bias as well. The efficiency can then be computed from the exposure, but neither the cost loss nor the residual loss ratios. Both remain under addressee's care, if he truly knows them. A connection to Section 7.1 is established here, that contributes to clarify the notion of *meta-decision* according to which a decision related to the fabrication of a decisional process is made.

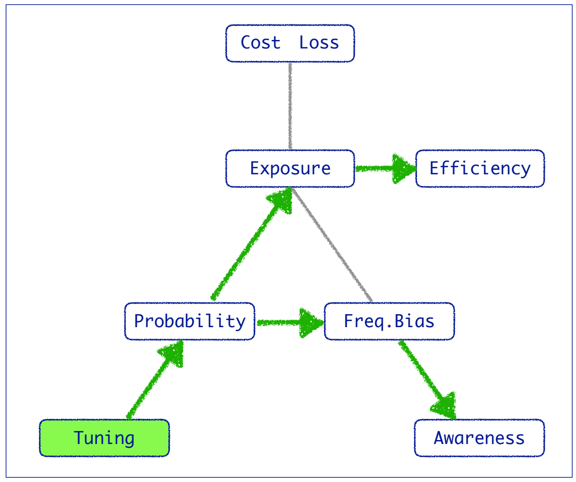


Figure 13: *Sagittal representation of the incidences occurring by the tuning of a warning system.*

Remembering the correspondence between frequency biases and exposures, by which high frequencies are linked to low exposures and inversely, we now develop this idea further. Having a definition of the exposure expressed in terms of probability of occurence and using equation (19) in Section 9, we notice that the tools required to formalize this relationship are now available. Following Figure 14 depicts this correspondence. It appears that the warning system may be tuned either by the choice of a frequency bias that eventually fixes the exposure or, conversely, by the choice of an exposure that determines the frequency bias. In this perspective, the probability threshold plays the role of an ancillary connector between both other factors. Both flows are described in Figure 15.

On Figure 14, the thin vertical line divides the area in sub domains where the frequency bias is greater, respectively lower than one. It crosses the frequency bias curve at probability $p = 0.65$ corresponding to $FB = 1$ and is in accord with the values shown in Figure 7. This clarifies the notion of risk awareness proposed in Section 7.2.

As shown in Figure 15, upper panel, only a knowledge of the cost & residual loss ratios allows us to cover the whole network. This information

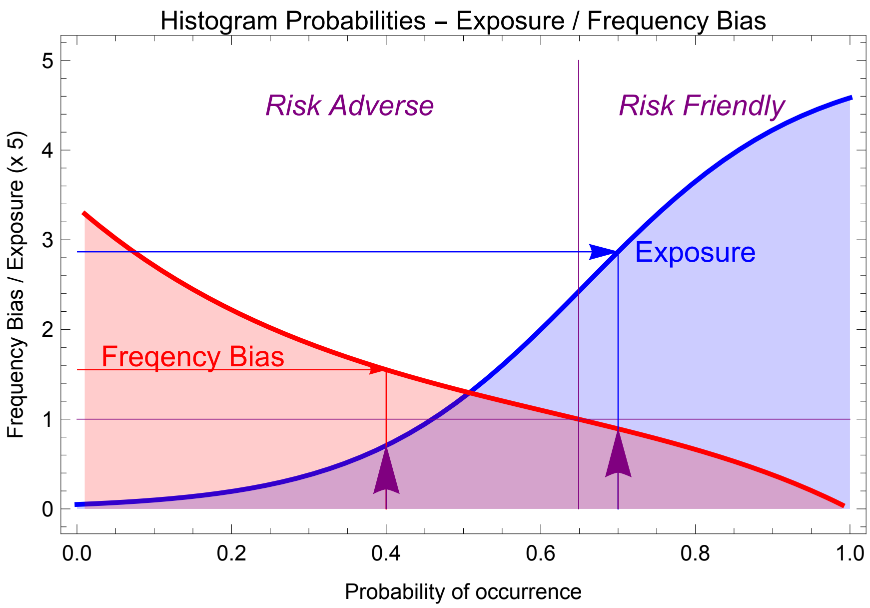


Figure 14: *Exposure and frequency bias (blue, respectively red curves). The warning system may be either tuned by the choice of a frequency bias that settles the exposure, or by the choice of an exposure that determines the corresponding frequency bias (vertical purple arrows). The thin vertical line divides the area in sub domains where the frequency bias is greater, respectively lower than one. This picture depicts the reciprocity firstly evoked in Section 8.2, Table 6.*

must be provided by the addressee, or might be guessed. Both ratios ought to be related to the intensity of adverse events and thus be formulated in dependency to to the intensity threshold Q.

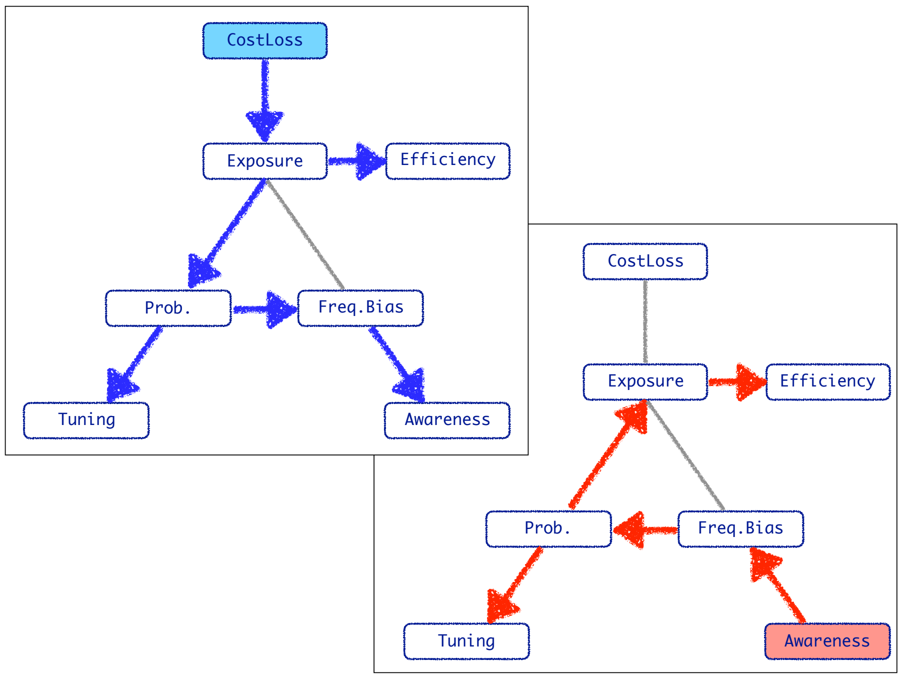


Figure 15: *Tuning of the warning system. The warning system may be either tuned by the choice of a frequency bias that eventually settles the exposure (lower panel) or, conversely, by the choice of an exposure that determines the frequency bias (upper panel).*

Our next challenge will consist in exploring the dependency on events occurring at and beyond various intensity thresholds. How shall we manage this? By introducing a new dimension to our domain and considering the space {Intensity, Probability, Exposure}.

11 Intensity, Probability, Exposure

Two expressions of the exposure are now available, where the former presents the addressee's viewpoint, the latter the issuer's. They are repeated below with indexes "a" for addressee and "i" for issuers:

$$\begin{aligned} E_{a(Q)} &= \frac{\Gamma_{(Q)}}{1-\Lambda_{(Q)}} \\ E_{i(p)} &= \frac{1}{1+\frac{v_{(p)}}{u_{(p)}}}|_Q. \end{aligned}$$

The first relation depends on the intensity threshold Q and is independent from the probability threshold p. The second one depends on the probability threshold p at intensity threshold Q. Making use of the efficiency $W_{(H,Far)}$, we have shown that $E_{i(p)} = E_{a(Q)}$ is a prerequisite for the optimal tuning of the warning system. Now, pragmatically looking for solutions $p = p^*{}_{(q)}$ of the equation for every $q \in [0, B]$, we address the minimal problem[6].

$$p^*{}_{(q)} \rightarrow \min_{p\in(0,1)} |E_{i(p)} - E_{a(q)}|, \ \forall q \in [0, B] \tag{24}$$

To this purpose, following to the path described in the upper left diagram of Figure 15, we firstly arbitrarily define an addressee's exposure $E_{a(q)}$ in relation to the intensity threshold Q.

Presented in Figure 16, this putative profile is similar to the other one shown earlier in the inbox of Figure 4. The Figure is 3-dimensional with the abscissa expressing the intensity of the event, the ordinate the probability, and the exposure in the vertical axis. The exposure generally raises as intensity increases with sharper steps sketched at intensities $Q = 0.35$ and $Q = 0.55$ suggesting thresholds where harmful disasters are likely to occur. The independency to the probability is made clear by the "flatness" of the surface in its ordinate direction. (On this diagram, as well as on all following figures, the meteorological intensity, represented in abscissa, is normed to one. The two other parameters, probability and exposure, are normalized to one by definition).

[6]Direct solutions exist: if $u_{(p)} = e^{\alpha p}$ and $u_{(p)} = e^{-\beta p}$, then $p^*{}_{(Q)} = \frac{1}{\alpha+\beta}\log(\frac{1}{E_{a(Q)}} - 1)$.

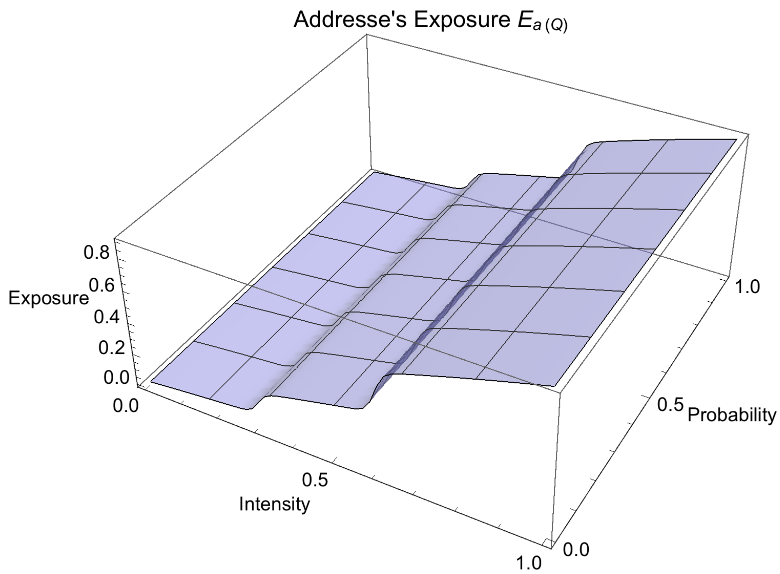


Figure 16: *Addressee's exposure expressed in terms of intensity (abscissa) and probability of occurence (ordinate), of an event. The addressee's exposure is explicitly independent from the probability of occurence.*

Considering now the issuer's side $E_{i(p)}$, we dare a solid assumption:

Assumption: taking into account experiences collected in meteorology, we establish a negative relationship between the intensity of the events having to be forecasted and the performance of the warning system, whose skill is supposed to decrease as events are becoming stronger and increasingly challenging. This is, beside the rationality expectation formulated in Section 2, our second working hypothesis.

Other assumptions might have been considered, as for example the stability of the warning performance under increasing meteorological intensity, or an assumption related to the temporal dependance on the deadline of the forecast, by which short time forecasts would be deemed more reliable than long term forecasts. Such assumptions could be easily integrated in the Mathematica code developed in parallel to the present essay.

The assumption shapes the issuer's exposure shown in Figure 17: the steepness of the function in its ordinate direction decreases as intensity increases. Each Section at a given intensity threshold Q, sketched as a "parallel" curve in the surface, bears the sigmoidal profile of the exposure seen in Figures 12 and 14. The other isolines associated to the reddish - bluish hue are iso - frequency bias lines. Highest frequency biases occur near the reddish lower right corner of the surface, lowest biases at the upper right bluish corner. The acuteness of the signal decreases towards left, where harmful events for which the the issuer's skill is considered better are represented.

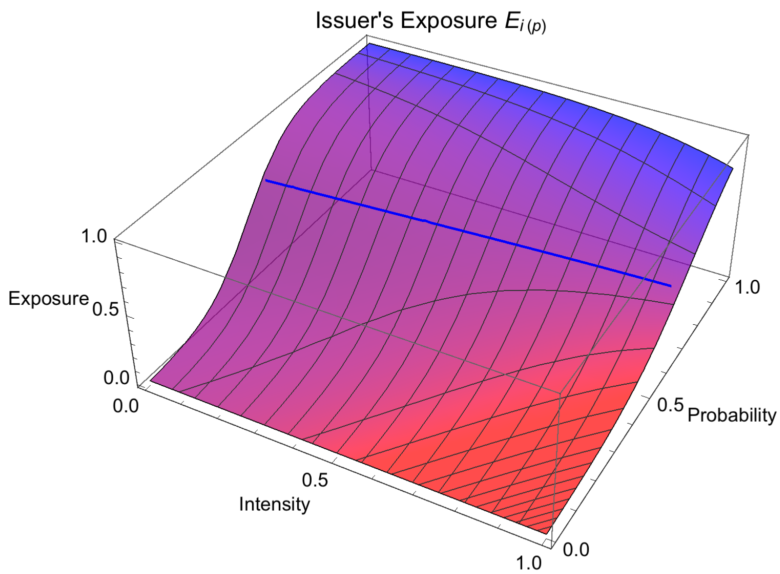


Figure 17: *Issuer's exposure expressed in terms of intensity (abscissa) and probability of occurence (ordinate) of an event. The hue, sustained by the isolines drawn in the surface, expresses the frequency bias at each intensity & probability.*

Summarizing so far, we notice that high frequency biases occur for events that are deemed intense and for which bulletins are delivered at low probability of occurence[7]. In any cases, the biases are induced by the issuer's communication strategy and formally not (directly) related to addressees' expectations.

[7]Sounds familiar? Should the reddish right corner be christened the "Hype Lagoon?"

11.1 Addressee and issuer

We are now ready to confront the two profiles[8]. This operation is completed in Figure 18 which is a merge of Figures 16 and 17. The interSection of the two surfaces, indicated as the curvy yellow line, is a geometrical representation of the operation defined in the expression (24): ${p^*}_{(q)} \rightarrow \min_{p\in(0,1)} \ |E_{i(p)} - E_{a(q)}|, \ \forall q \in [0, B]$. It visualizes the optimal warning decision as a relation between the intensity threshold Q and the corresponding probability threshold ${p^*}_{(Q)}$.

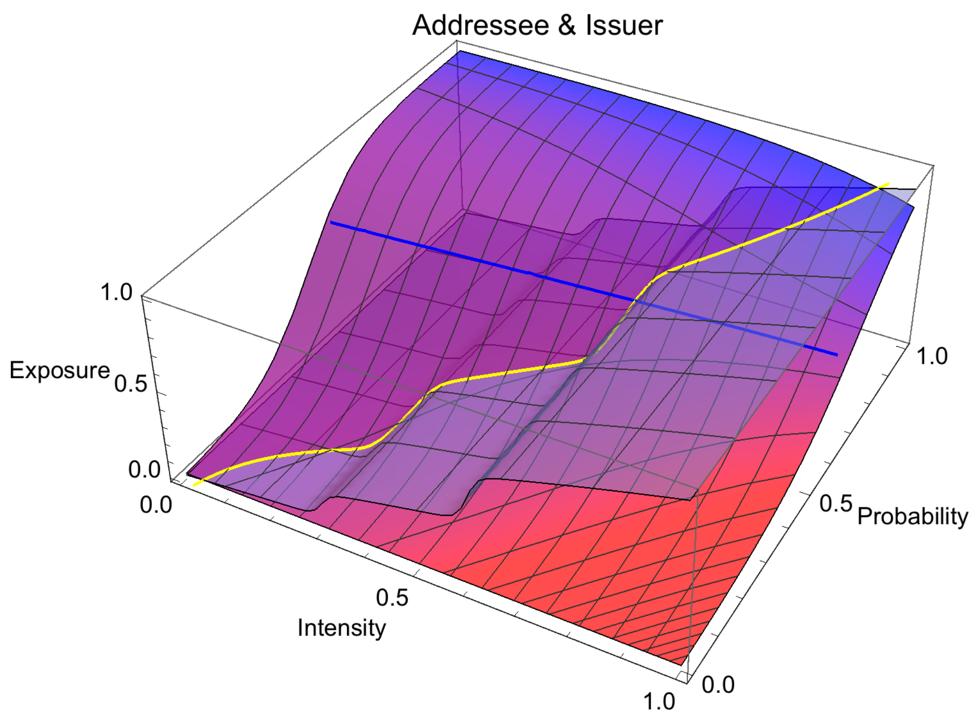


Figure 18: *Addressee's and issuers's exposures corresponding to Figures 16 and 17. Abscissa: normalized intensity of the event. Ordinate: probability. Vertical dimension: exposures. The optimal probability thresholds are pictured in yellow, they correspond to the interSection of both surfaces. The blue curve depicts the neutral frequency bias.*

[8]Pictured as title image in accordance to an old Chinese proverb "a single drawing is thousand words worth".

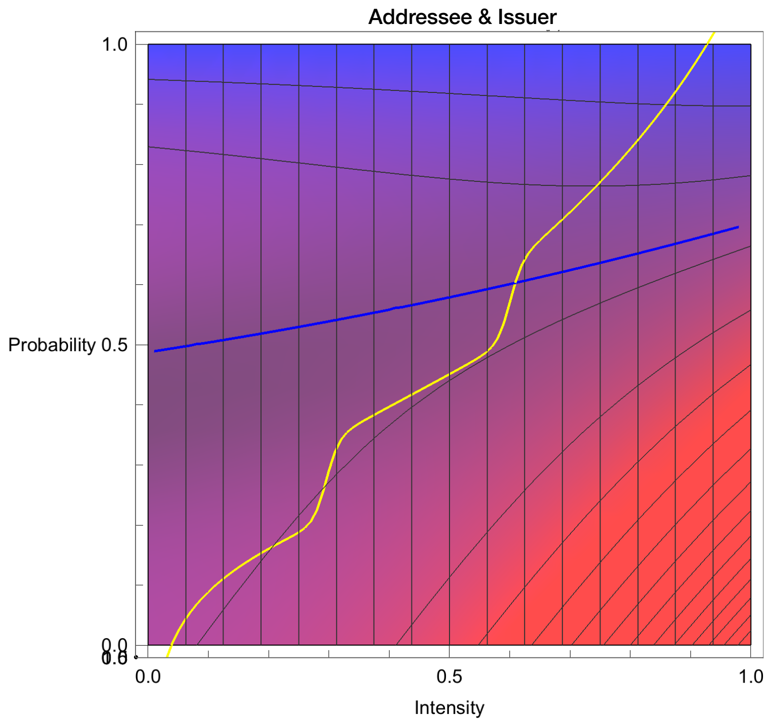


Figure 19: *Orthographic projection of Figure 18. Abscissa: normalized intensity of the event. Ordinate: probability. The optimal probability threshold is pictured in yellow as a function of the intensity. The black curves are iso - frequency bias lines. To the blue curve corresponds the neutral frequency bias. The frequency bias is maximum near the lower right corner.*

The optimal probability threshold increases as the intensity (as well as the addressee's exposure) increases, nicely undulating along the iso - frequency bias line $FB = 1.12$ until a normalized intensity $Q \approx 0.6$. It diverges beyond this value. At the same time, the corresponding frequency bias decreases below one ($FB < 1$). Due to high exposures, mitigating costs become similar to disaster costs: accomplished addressees are likely to disregard warnings delivered under such auspices. The inclusion of the efficiency in the analysis enables a quantification the issue.

11.2 Inclusion of the efficiency

Figure 20 is the identical to Figure 17, its orthographic projection is shown in Figure 21. Curves drawn in the surface are now iso-efficiency lines and iso-frequency lines. The three blue curves represent the frequency bias 1.12, below, 1.0 middle curve and 0.88 upper curve. To the green iso-curve corresponds an efficiency $W = 0.4$. The efficiency is higher than this value within the area encompassed by the curve, lower outside. The locus of probability thresholds is figured as the yellow curve, as in Figure 18. It almost follows the iso frequency bias curve $FB = 1.12$ until a point $\{Q \simeq 0.65, p^*{}_{(Q)} \simeq 0.63\}$ and increases sharply beyond.

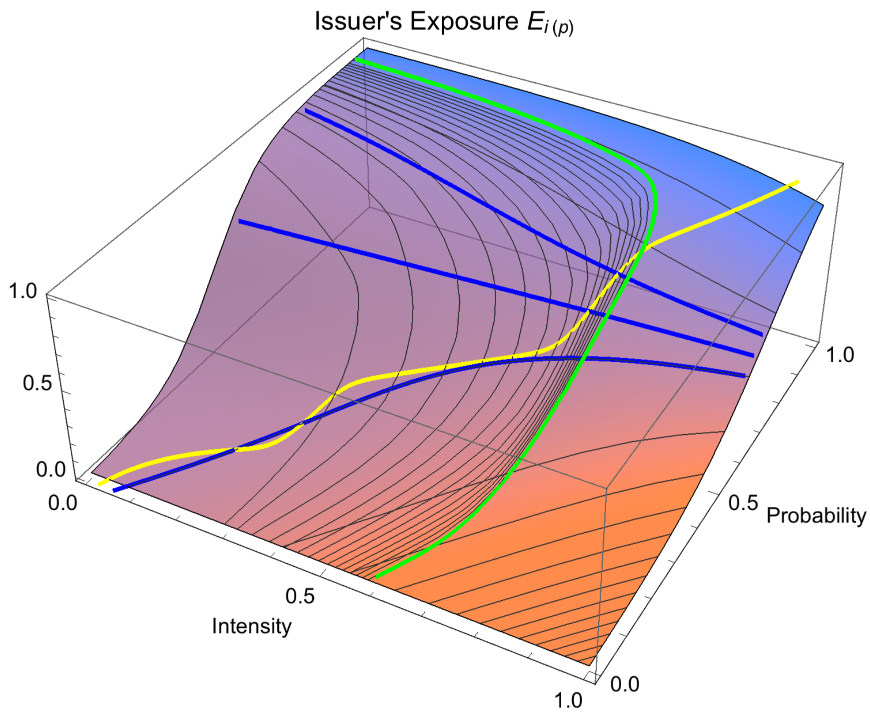


Figure 20: *Issuer's exposure with iso-efficiency lines and iso-frequency lines.*

To the area encompassed by the iso-frequency bias curves correspond reasonable tuning values of a warning system. In this drawing, the isoline $FB = 1.12$ has been chosen in order to follow approximately the locus of optimal probabilities, and the other isoline $FB = 0.98$ symmetrically to the neutral isoline $FB = 1.0$. To low probabilities and weak intensities, at the left lower corner, correspond usual weather bulletins diffused on a daily basis. The right lower corner brands avalanches of bulletins delivered at low probabilities for intense events by potentially unskilled forecasters (according to the assumption made earlier).

For intense events occurring at normalized intensities larger than 0.75, the probabilities required to trigger warnings reach unsustainable values. Under such conditions, the addressee is likely to be almost never warned anymore and thus required to relay on insurances services. The other option at his disposal would consist in taking physical, protective actions aimed at modifying his exposure profile.

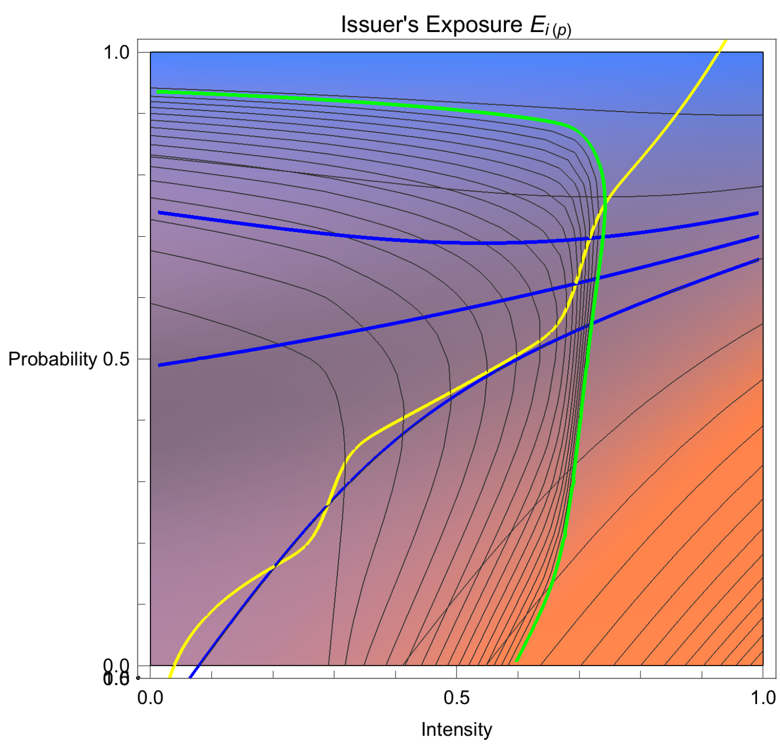


Figure 21: *Orthographic projection of Figure 20. Two networks are pictured: the iso frequency bias curves, with three of them underlined in blue at values* $FB = \{1.12, 1.0, 0.88\}$*. and the network of iso-efficiencies curves. Efficiency is greater than* 0.4 *inside the area encompassed by the green iso-efficiency curve and lower outside.*

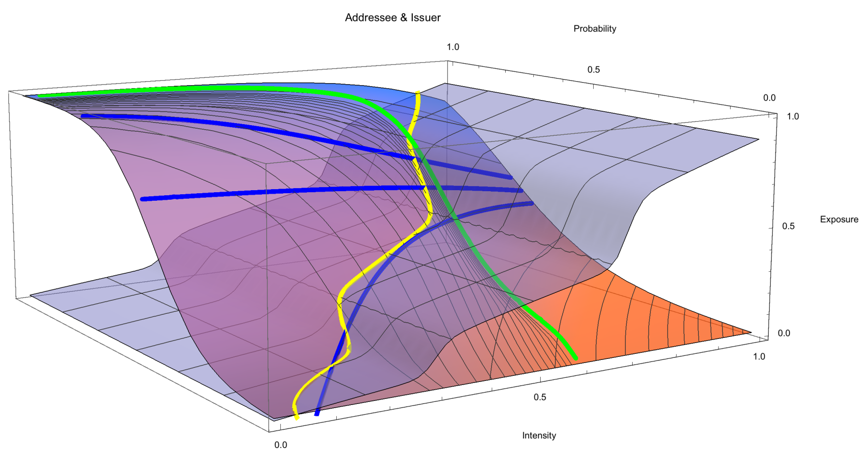


Figure 22: *Other perspective of Figures 18 and 20. All elements discussed so far are integrated. The horizontal axes represent the intensity of a weather event and its probability of occurence. The addressee's and issuer's exposures are depicted as surfaces deployed in the vertical dimension. The curves imbedded in the issuer's surface are related to the efficiency of the warning system with the green boundary, and to the frequency bias in blue. The optimal decision is figured as the interSection of both exposure surfaces, in yellow.*

11.3 Creatio ex nihilo: virtual addressee

Following Figure 23 suggests an answer to the issue evoked of a modification of the addressee's exposure and presents at the same time the operation suggested in the lower right panel of Figure 15.

The choice of a frequency bias, in this case $FB = 1.12$, determines the setting of the warning system: the optimal probability threshold now follows the frequency bias isoline. This process implicitly defines the exposure of a new, virtual addressee, whose exposure culminates at $E_{a(Q=0.78)} = 0.38$. The efficiency pattern is transformed in accordance to the new exposure. Its boundary $W = 0.45$ is drawn in green on the figure.

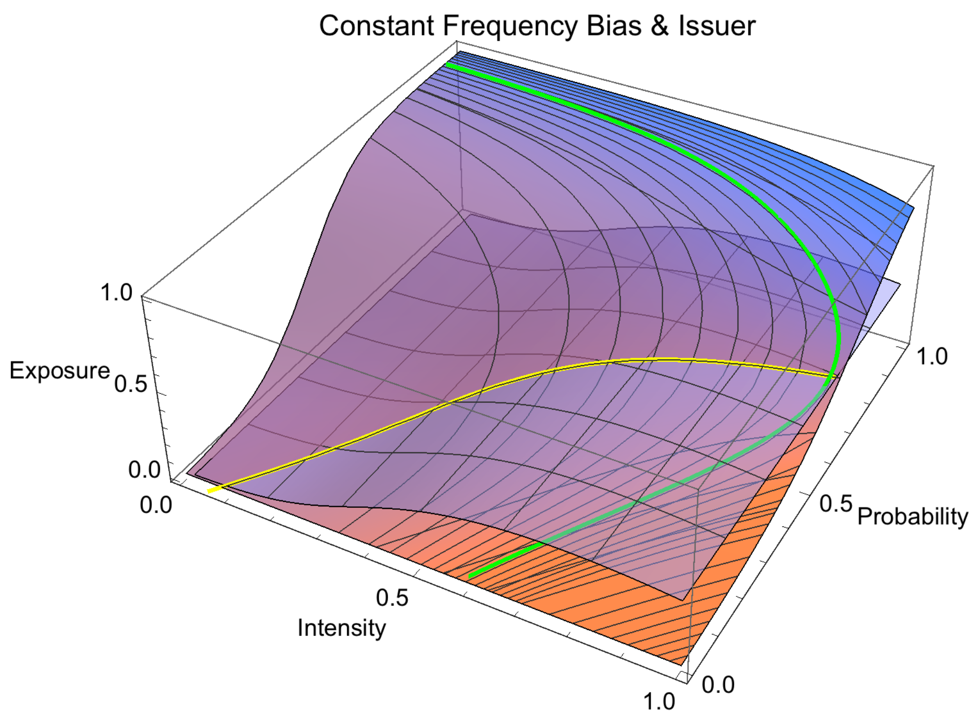


Figure 23: *Setting of a warning system tuned according to a constant frequency bias $FB = 1.12$ for each intensity threshold Q. This settles the probability threshold $p^*_{(Q)}$ shown in yellow on the issuer's profile and defines the addressee's exposure. The efficiency pattern is transformed accordingly, with its efficiency boundary $W = 0.45$ drawn in green.*

Of course, different profiles of virtual addressees may be conceived according to frequency bias values chosen to be dependent on the intensity, or related to the efficiency of the warning system. In each cases, attention has to be dedicated to the loop that is induced: establishing a exposure profile related to the frequency bias or the efficiency has an impact on both of these parameters.

11.4 Performance improving with intensity

We now reverse the assumption made at the beginning of this section and consider a warning system whose performance increases with the intensity of the meteorological events. Otherwise, all settings remain unchanged with the addressee's profile depicted in Figure 16.

The sigmoid shape of the issuer's function is now flipped in Figure 24 with the "lazy" shape laid at low intensities on the left side of the diagram and the sharper shape at high intensities, on the right side. An orthogonal projection is shown in Figure 25. The underlying hue depicts the frequency bias, as in Figure 19. However, the frequency bias is maximum at low intensities, now corresponding to low forecasting skills.

The yellow curve of optimal probabilities, is recomputed according to this new setting and one notices that probability thresholds remain reasonable, and do no longer raise to values as high as in Figure 18.

The striking element, however, is provided by the isolines of the efficiency function, Equation (16), drawn in the surface. The efficiency reaches local maxima located closely to the left of the exposure thresholds featured on Figure 16. This indicates that warnings are optimally delivered at intensities slightly lower than the exposure thresholds relevant to the addressee, with the jumps of probability thresholds located within these efficiency maxima. This property, nicely emerging from the theoretical setting of this work, is well known form practitioners in the field of environmental risk management.

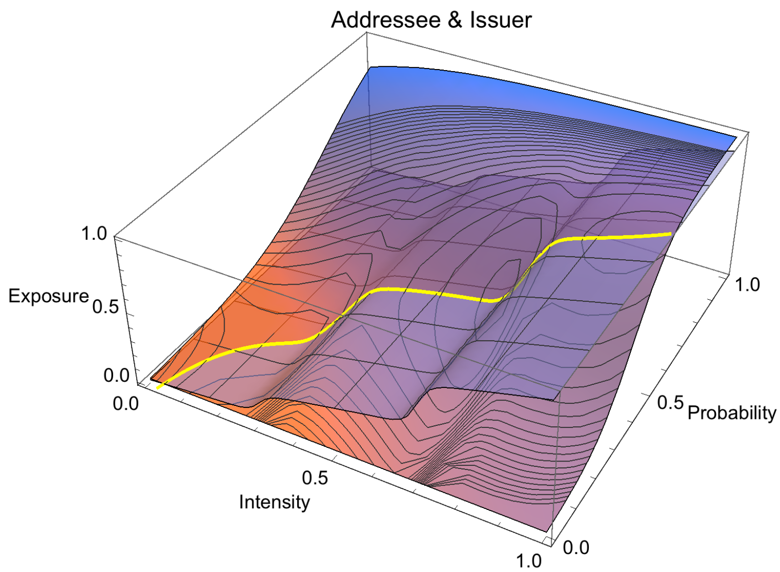


Figure 24: *Setting of a warning system operated following a forecasting model whose performance increases with the intensity of the meteorological events. The isolines drawn in the surface are iso-efficiency lines. Local minima are settled at intensities located slightly below the intensity thresholds relevant for the addressee.*

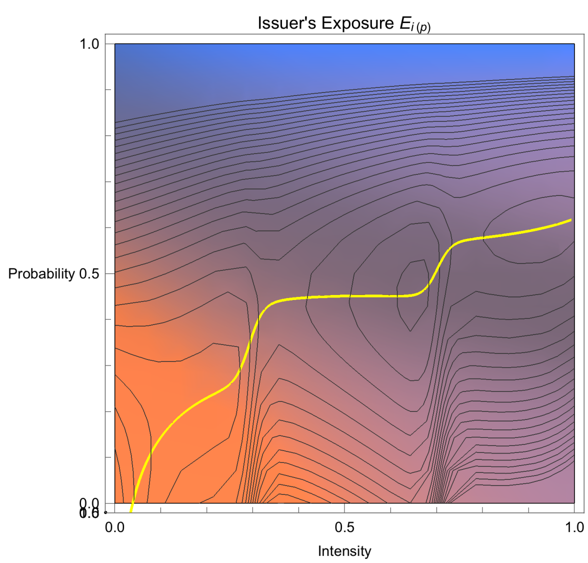


Figure 25: *Orthogonal projection of the previous Figure.*

11.5 Forecaster and freewill

Who decides exerts his/her freewill. Expressed in our context of warning optimality, this sentence can be interpreted according to two points of view:

If the decider, who for us is a forecaster, freely seeks optimality while taking warning decisions, then she accepts to be guided by the schemes presented in this work, or by others schemes of similar quality. The criteria she is required to follow emanate on one side from the properties of the warning system, on the other side from the addressee's exposure. For her are these elements exogenous in their essence, with the consequence of letting the room for her freewill dwindling. But, of course, she may also decide to keep her freewill open, thus disregarding exogenous criteria at the probable cost of sub-optimality.

Indeed are some personalities striving to attitudes keeping their freewill "whatever it takes" open, endorsing lacking objectivity resulting in sub-optimality. Other characters prefer to foster systemic optimality, at the cost of harnessing their own freewill. In such constellations, the freewill argument happens to become a meta-argument ultimately confined into a binary choice: to be optimal, or not[9].

Of course is this speculation gender independent, and even valid for a machine or for any algorithm implemented on a computer.

12 Modified Climate

All computations performed so far are climate independent: the climatic component Ω vanished during the derivation of the exposure, Equation (7) and in the definition of the efficiency, Equation (16). In this respect, the methodology developed so far proceeds as a climate-independent gauge of a warning system. However, as discussed in Section 10, a modification of the frequency distribution of weather events is likely to have an impact of the economic outcome of the addressee. Thus, in order to investigate this impact, we firstly need to simulate a "low" and a "high" climate, as presented in Figure 26, where the frequency of extreme events substantially increases under high climate conditions.

[9] Not only might William Shakespeare have liked, but certainly Arthur Schopenhauer too, whose magnum opus is titled "The World as Will and Representation".

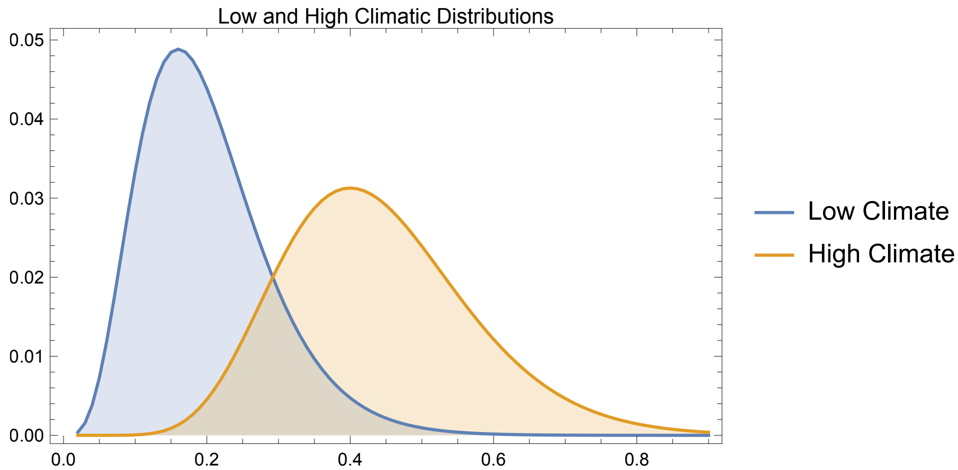


Figure 26: *Low and high climates are simply figured as probability distribution functions of a gamma distribution.*

We consider the fate of the addressee whose exposure is sketched in Figure 16 and evaluate the following:

$$Ratio = \frac{\text{High Climate } - \text{ Low Climate}}{\text{Low Climate}}$$

Three evaluations are performed for the indicators listed below according to the integral expression:

$$\int_0^Q \mathcal{X}_{(q)} C_{\{Low,High\}(q)}\, dq$$

1) $\mathcal{X}_{(q)} = E_{(q)}$: the integral measures the increase of the climate burden the addressee will be be submitted to under the "Low" and "High" climates.

2) $\mathcal{X}_{(q)} = \mathcal{W}_{(q)}$: the integral measures the evolution of the efficiency of the warning system as the climate evolves from "Low" to "High" ($\mathcal{W}_{(q)}$ is defined below):

$$\mathcal{W}_{(q)} = W_{(H_{(p^*_{(q)})}, Far_{(p^*_{(q)})})}$$

3) $\mathcal{X}_{(q)} = Eco_{(q)}$: the integral measures the relative economic efficiency. It is formed as ratio between the costs spared when the warning system is connected instead of being disconnected (climate burden minus actual

costs), divided by the costs induced when the warning system is disconnected (climate burden):

$$Eco_{(q)} = \frac{M_{A(0,0)} - M_{A(H_{(p^*_{(q)})}, Far_{(p^*_{(q)})})}}{M_{A(0,0)}}$$

The results are presented in the following Table 7.

	Low Climate	High Climate	Ratio	Table 7
Climatic Burden	0.003439	0.01369	+298 %	
Efficiency	0.4942	0.4617	-6.59 %	
Relative Economic	0.4607	0.3391	-26.39 %	

This very simple simulation shows several interesting phenomena. Firstly, the simulated climate change is really dramatic and has a startling incidence on the addressee's business, would he not make use of the alarms delivered by the warning system. Secondly, the warning system is quite resilient to the climatic change: its efficiency decreases only by about 6.6% in this simulation. This abatement is mainly provoked by the fact that the warning system has been intentionally tuned in order to become less performant when confronted with pugnacious meteorological events. The evolution of the relative economic efficiency measures the weakening of the warning performance, as well as the increasing addressee's vulnerability under harsh climatic conditions.

Under such circumstances, two actions might be considered, the first one practical, the second theoretical. From a practical point of view, the addressee must imperatively modify his exposure in oder to increase his resilience towards new climatic conditions. On the theoretical side, The definition of the probabilistic threshold $p^*_{(q)}$ should be evaluated in accordance to the frequency of occurrence of the weather events according to a full fledged variational scheme. This task will not be undertaken within the frame of this essay.

13 Conclusions

The scheme presented in this essay is conceptual. It is based on an elementary cost-loss model that, despite its simplicity, names and describes key factors influencing rational decisions. These factors are the intensity of an incoming weather event, its probability of occurence and the exposure - or vulnerability - of a person or an agent whose activity is susceptible to be impaired by this event.

The methodological approach distinguishes the roles of the issuer of a warning and the addressee of this warning. Although sharing some characteristics with the game theory, the setting differs from a classical game in the sense that both actors are not competing to acquire the largest part of common ressource.

The Relative Operation Characteristic is based on ratios and emancipates itself from strict financial considerations. All parameters introduced, such those describing threats, or intensity, vulnerability, or exposure, efficiency, are expressed as ratios in a non-monetary way. Furthermore, the efficiency, or impact, is designed as a climate independent gauge of a warning system.

The Frequency Bias happens to be connected with the others parameters in a subtle manner and discloses the issuers's communication custom. It enables a formulation of the risk awareness that is different, although connected to the notion of exposure, or vulnerability.

The method enables the elaboration of decisions based on probabilistic data emanating from ensemble forecasting systems, bayesian, neural or genetic algorithms. It suggests the definition of virtual addressees whose profiles are optimally suited to the aforementioned algorithms.

As promised is Section 2, Aims, most of the key concepts developed so far have been presented as geometric or graphical structures.

Figure 27 summarizes these elements and localizes the deployment of a decision process within the sequential chain of a warning system.

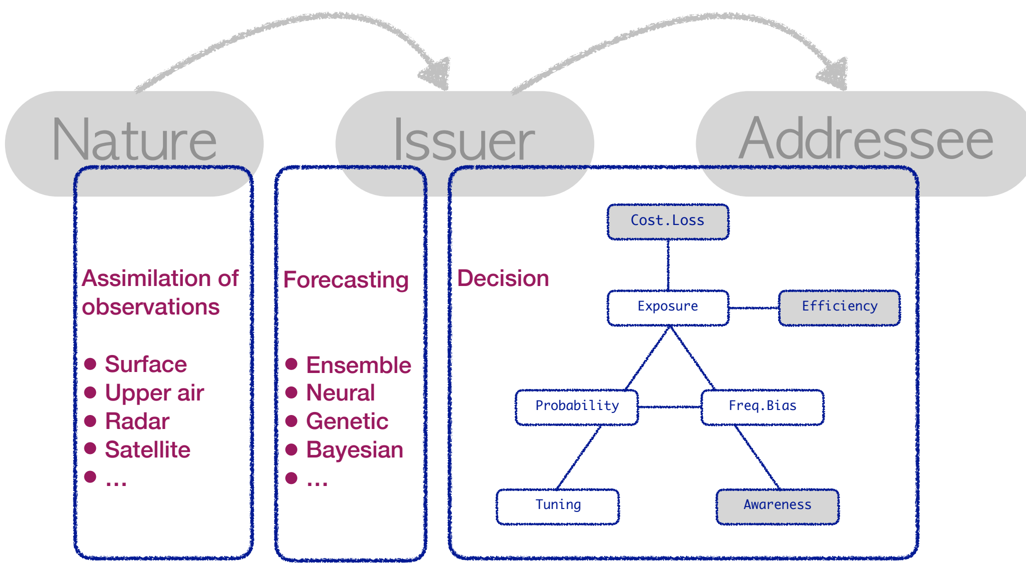


Figure 27: *Information flow and localization of the decision process in the operation chain of a warning system.*

Acknowledgement

My thanks are addressed to to all the colleagues with whom I experienced active interactions over the last years of my professional work history at MeteoSwiss.

Author's Address

Jacques Ambühl
Grosswiesenstrasse 167
CH-8051 Zürich

ambuhl@icloud.com

https://www.visualambuhl.ch

14 Annexes

14.1 Derivation of the economic function

The derivation is performed according to the Table(4) defined in Section 5 and the basic definitions provided in Section 3.

14.1.1 Derivation of $M_{A(H,Far)}$

Making use of the substitution *ad infinitum* of $c = Far\ (c+d)$ into itself, one M_A is expressed as[10]:

$$
\begin{aligned}
M_A &= \frac{1}{a+b+c+d}\,[bL + Cc + (C+\lambda)d] \\
&= \frac{\Omega}{b+d}\,[bL + d\lambda + C(c+d)] \\
&= \Omega\,[\frac{b+d-d}{b+d}L + \frac{d}{b+d}\lambda + C\frac{c+d}{b+d}] \\
&= \Omega\,[(1-H)L + H\lambda + C\frac{Far\ (c+d)+d}{b+d}] \\
&= \Omega\,[(1-H)L + H\lambda + C(H + Far\ \frac{c+d}{b+d})] \\
&= \Omega\,[(1-H)L + H\lambda + C(H + Far\ \frac{Far\ (c+d)+d}{b+d})] \\
&= \Omega\,[(1-H)L + H\lambda + C(H + HFar + Far^2\ \frac{c+d}{b+d})] \\
&= \Omega\,[(1-H)L + H\lambda + C(H + HFar + HFar^2 + Far^3\ \frac{c+d}{b+d})] \\
&= \lim_{n\to\infty}\Omega\,[(1-H)L + H\lambda + C(H + HFar + HFar^2 + \cdots + Far^n\ \frac{c+d}{b+d})] \\
&= \Omega\,[(1-H)L + H\lambda + \frac{CH}{1-Far}].
\end{aligned}
$$

The geometric sequence (with the frequency bias $\eta = \frac{c+d}{b+d}$) is easily identified:

$$S_n = H + H\ Far + H\ Far^2 + H\ Far^3 + \cdots + H\ Far^{n-1} + \eta\ Far^n.$$

[10] Although I could have made direct use of the definition of the frequency bias, I loved this alternative derivation.

It satisfies:

$$S_n\ (1 - Far) = H + (\eta - H)\ Far^n - \eta\ Far^{n+1}$$

and, therefore:

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{1}{1 - Far}(H + (\eta - H)\ Far^n - \eta\ Far^{n+1}) \\
&= \frac{H}{1 - Far} \qquad with\, Far < 1 \\
&= FB.
\end{aligned}$$

The convergence of the infinite sequence occurs only for $Far < 1$, a requirement which can be expected to hold for the false alarm ratio. The above expression can be rearranged in a scalar product as follow:

$$\begin{aligned}
M_{A(H,Far)} &= \Omega\,[(1-H)\,L + H\,\lambda + \frac{C\ H}{1 - Far}] \\
&= L\,\Omega\,[(1-H) + H\,\frac{\lambda}{L} + \frac{C}{L}\,\frac{H}{1 - Far}] \\
&= L\,\Omega\,[(1-H) + H\,\Lambda + \Gamma\,\frac{H}{1 - Far}] \\
&= L\,\Omega \cdot \begin{bmatrix} 1 & \Lambda & \Gamma \end{bmatrix} \boldsymbol{.} \begin{bmatrix} 1 - H \\ H \\ H\,(1 - Far)^{-1} \end{bmatrix}.
\end{aligned}$$

14.1.2 Derivation of $M_{R(H,F)}$

$$\begin{aligned}
M_R &= \frac{1}{a+b+c+d}[bL + C\ c + (C + \lambda)d] \\
&= \frac{1}{a+b+c+d}[Cc + d(C + \lambda - L) + (b+d)L] \\
&= C\frac{c}{a+c} \cdot \frac{a+c}{a+b+c+d} + \frac{d}{b+d} \cdot \frac{b+d}{a+b+c+d}(C + \lambda - L) \\
&\quad + \frac{b+d}{a+b+c+d}\ L \\
&= C\ F\frac{a+b+c+d-b-d}{a+b+c+d} + H\ \Omega\ (C + \lambda - L) + \Omega\ L \\
&= C\ F\ (1 - \Omega) + H\ \Omega\ (C + \lambda - L) + \Omega\ L \\
&= L\ [\Gamma\ F\ (1 - \Omega) + H\ \Omega\ (\Gamma + \Lambda - 1) + \Omega].
\end{aligned}$$

14.2 The Implicit Function Theorem

Let an open set $A \subset R^{n+k}$ and let $f : A \to R^n$ be a C^r function. Write f in the form $f(x,y)$, where $x \in R^k$ and $y \in R^n$. Suppose that the point $(a,b) \in A$ such that $f(a,b) = 0$ and the determinant of the $n \times n$ matrix whose elements are the derivatives of the n component functions of f with respect to the n variables, written as y, evaluated at (a,b), is not equal to zero. The latter may be rewritten as $Rank[(Df(a,b))] = n$.

Then there exists a neighborhood B of a, $B \subset R^k$, and a unique C^r function $g : B \to R^n$ such that $g(a) = b$ and $f(x, g(x)) = 0$ $\forall x \in B$.

Computing the total differential of $f(x, g(x))$ with respect to x:

$D_x f(x, g(x)) = \partial_x f(x, g(x)) + \partial_y f(x, g(x)) \, \frac{dg}{dx} = 0$, yields:

$\frac{dg}{dx} = -(\partial_y f(x, g(x)))^{-1} \, \partial_x f(x, g(x))$, as applied in the main text.

Source: http://mathworld.wolfram.com/ImplicitFunctionTheorem.html.

14.3 Integration of a differential equation

The equation is expressed as:

$$\frac{dH}{H} = \frac{E}{(F-1)(E+F-1)} \, dF = (\frac{1}{F-1} - \frac{1}{E+F-1}) \, dF$$

integrating to $\log(H) = \log(F-1) - \log(E+F-1) + c$, thus delivering:

$$H_{(F)} = C \, \frac{F-1}{E+F-1}$$

with the integration constant C expressed as $\log(c)$. One notices the vertical asymptote located at $E = 1 - F$ that is relevant in the main text.

14.4 Exposures

14.4.1 Derivation of $\frac{1}{1+\frac{v_{(p)}}{u_{(p)}}} = E_{(Q)}$

Starting with Equation (21) and making use of the definitions provided in Section 8, we compute $\partial_p \left[W_{(H_{(p)}, Far_{(p)})} \right] = 0$:

$$\begin{aligned} \partial_p \left[W_{(H_{(p)}, Far_{(p)})} \right] &= \frac{1}{1 - E_{(Q)}} (\partial_p H_{(p)} - \partial_p FB_{(p)}\ E_{(Q)}) \\ \partial_p H_{(p)} &= \partial_p FB_{(p)}\ E_{a(Q)}\ ;\ E_{(Q)} \neq 1. \\ \frac{1}{\Pi} u_{(p)} &= \frac{E_{(Q)}}{\Pi} (u_{(p)} + v_{(p)})\ ;\ \Pi > 0. \\ \frac{1}{1 + \frac{v_{(p)}}{u_{(p)}}} &= E_{(Q)}. \end{aligned}$$

14.4.2 Derivation of $E_{(p)} = \frac{u_{(p)}}{u_{(p)}+v_{(p)}}$

The differential equation (15), repeated below, is firstly expressed in terms of E:

$$\frac{\partial H}{\partial Far} = \frac{E\, H}{(Far - 1)(E + Far - 1)}$$

$$E_{(p)} = \frac{(Far - 1)^2}{H \frac{\partial Far}{\partial H} - Far + 1} = \frac{(Far - 1)^2}{(\frac{1}{H}\frac{\partial H}{\partial p})^{-1} \cdot \frac{\partial Far}{\partial p} - Far + 1}$$

As all terms present in the right hand side are function of the probability p, it is legitimate to express the exposure as $E_{(p)}$. Let us now compute its components, firstly the complement to one of the false alarm ratio at a probability threshold p:

$$Far - 1 = \frac{\int_p^1 v_{(\pi)} d\pi - \int_p^1 (u_{(\pi)} + v_{(\pi)}) d\pi}{\int_p^1 (u_{(\pi)} + v_{(\pi)}) d\pi} = -\frac{\int_p^1 u_{(\pi)} d\pi}{\int_p^1 (u_{(\pi)} + v_{(\pi)}) d\pi},$$

- the derivative of the hit rate with respect to the probability threshold p:

$$\frac{1}{H}\frac{\partial H}{\partial p} = \frac{\int_0^1 u_{(\pi)} d\pi}{\int_p^1 u_{(\pi)} d\pi} \cdot \frac{-u_{(p)}}{\int_0^1 u_{(\pi)} d\pi} = -\frac{u_{(p)}}{\int_p^1 u_{(\pi)} d\pi},$$

- the derivative of the false alarm ratio with respect to the probability threshold p:

$$\begin{aligned}
\frac{\partial Far}{\partial p} &= \frac{-v_{(p)}\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi + (u_{(p)}+v_{(p)})\int_p^1 v_{(\pi)}d\pi}{(\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi)^2} \\
&= \frac{\int_p^1 [-u_{(\pi)}v_{(p)} - v_{(\pi)}v_{(p)} + u_{(p)}v_{(\pi)} + v_{(p)}v_{(\pi)}]d\pi}{(\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi)^2} \\
&= \frac{u_{(p)}\int_p^1 v_{(\pi)}d\pi - v_{(p)}\int_p^1 u_{(\pi)}d\pi}{(\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi)^2}.
\end{aligned}$$

Introducing those partial results into the formulation of the exposure E, one gets:

$$\begin{aligned}
E_{(p)} &= \frac{(Far-1)^2}{(\frac{1}{H}\frac{\partial H}{\partial p})^{-1}\cdot\frac{\partial Far}{\partial p} - Far + 1} \\
&= \frac{(\int_p^1 u_{(\pi)}d\pi)^2}{(\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi)^2[-\frac{\int_p^1 u_{(\pi)}d\pi}{u_{(p)}}\cdot\frac{u_{(p)}\int_p^1 v_{(\pi)}d\pi - v_{(p)}\int_p^1 u_{(\pi)}d\pi}{(\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi)^2} + \frac{\int_p^1 u_{(\pi)}d\pi}{\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi}]} \\
&= \frac{\int_p^1 u_{(\pi)}d\pi}{(\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi)^2[-\frac{1}{u_{(p)}}\cdot\frac{u_{(p)}\int_p^1 v_{(\pi)}d\pi - v_{(p)}\int_p^1 u_{(\pi)}d\pi}{(\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi)^2} + \frac{1}{\int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi}]} \\
&= \frac{\int_p^1 u_{(\pi)}d\pi}{-\int_p^1 v_{(\pi)}d\pi + \frac{v_{(p)}}{u_{(p)}}\int_p^1 u_{(\pi)}d\pi + \int_p^1 (u_{(\pi)}+v_{(\pi)})d\pi} \\
&= \frac{\int_p^1 u_{(\pi)}d\pi}{\frac{v_{(p)}}{u_{(p)}}\int_p^1 u_{(\pi)}d\pi + \int_p^1 u_{(\pi)}d\pi} \\
&= \frac{1}{1+\frac{v_{(p)}}{u_{(p)}}} = \frac{u_{(p)}}{u_{(p)}+v_{(p)}}.
\end{aligned}$$

The condition $\int_p^1 u_{(\pi)}d\pi > 0$ has to be satisfied. It specifies that the system actually delivers warnings at probabilities $\geq p$. Otherwise, no hypothesis has been made that would specify the shape of the $u_{(p)}$ and $v_{(p)}$ distributions.

I discovered this derivation first, early in the development of my ideas, and found the previous one, related to the efficiency, later.

15 References

1. World Meteorological Organization. Public Weather services. Verification.

2. Richardson D.S. 2000. Skill and economic value of the ECMWF ensemble prediction system, (Q.J.R. Met. Soc. 2000; 126, pp. 649-667)

3. Jolliffe, I.I. and D.B. Stephenson, 2003. Forecast verification: A Practitioners Guide in Atmospheric Sciences (Wiley, pp. 66 - 76)

4. Nurni P. 2003. Recommendations on the Verification of local weather forecasts. (European Centre for Medium-Range Weather Forecasts; Technical Memorandum 430)

5. Ambühl, J. 2010. Customer Oriented Warning Systems, Veröffentlichungen der MeteoSchweiz, 84, 86 pp.

6. Ambühl, J. 2013. Le Dilemme de Cassandre. Revue MeteoMagazine. pp.58-61.

7. Ambühl, J. 2014. Kassandras Dilemma. IngenieurBiologie, Mitteilungsblatt Nr. 4/2014. pp.18-22.

8. Kube M, Ambühl J, Appenzeller C, Binder P, Comment T, Felix C, Fisler J, Frei C, Gaia M, Germann U, Grüter E, Hering A, Keller F, Mayoraz L, Meier D, Nisi L, Panziera L, Reiniger T, Romang H, Reynolds S, Schmutz C, Stoll M, Vogt S und Calpini B: 2016, Optimierung der Warnung und Alarmierung vor Naturgefahren bei MeteoSchweiz, Fachbericht MeteoSchweiz, 259, 136 pp.

9. Mayoraz L, Ambühl J: 2016. Automatic Gale Warning Proposals for Swiss Lakes and Regional Aerodromes, Scientific Report MeteoSwiss, 102, 70 pp. .